

Spectral curves and the mass of hyperbolic monopoles

Paul Norbury *

*Department of Mathematics and Statistics
University of Melbourne, 3010 Australia*

Nuno M. Romão †

*School of Mathematical Sciences
University of Adelaide, 5005 Australia*

December 2005

Abstract

The moduli spaces of hyperbolic monopoles are naturally fibred by the monopole mass, and this leads to a nontrivial mass dependence of the holomorphic data (spectral curves, rational maps, holomorphic spheres) associated to hyperbolic multi-monopoles. In this paper, we obtain an explicit description of this dependence for general hyperbolic monopoles of magnetic charge two. In addition, we show how to compute the monopole mass of higher charge spectral curves with tetrahedral and octahedral symmetries. Spectral curves of euclidean monopoles are recovered from our results via an infinite-mass limit.

MSC (2000): 32L25, 14H70; PACS (2003): 14.80.Hv

1 Introduction

Magnetic monopoles are paradigmatic examples of topological solitons in gauged field theories. Once a metric on \mathbb{R}^3 has been fixed, a monopole is defined as a pair (d_A, Φ) satisfying the Bogomol'nyi equation

$$B_A = *d_A\Phi \tag{1}$$

*e-mail: pnorbury@ms.unimelb.edu.au

†e-mail: nromao@maths.adelaide.edu.au

modulo gauge equivalence. Here, B_A is the curvature of the $SU(2)$ -connection d_A on $\text{End } \mathcal{E}$, where $\mathcal{E} \rightarrow \mathbb{R}^3$ is the trivial vector bundle associated to the defining representation, and Φ (the Higgs field) is a section of $\text{End } \mathcal{E}$ with constant and nonzero norm at infinity,

$$\lim_{|\mathbf{x}| \rightarrow \infty} \|\Phi(\mathbf{x})\| = m > 0. \quad (2)$$

In the radial compactification of \mathbb{R}^3 , the boundary condition (2) provides a map

$$\Phi|_{S_\infty^2} : S^2 \rightarrow S^2$$

from the 2-sphere at infinity to the 2-sphere of radius m centred at the origin of $\mathfrak{su}(2) \cong \mathbb{R}^3$, whose degree k is interpreted as the magnetic charge of the monopole. The Bogomol'nyi equation (1) turns out to be quite tractable when the metric has constant curvature, for then twistor methods can be used to characterise the solutions in terms of objects from complex geometry (holomorphic bundles, spectral curves, rational maps). There is an extensive literature focused on the case where the metric is euclidean. Then the space of all monopoles with a given topology carries a natural L^2 metric which is hyperkähler, and its geodesic flow can be interpreted physically as slow motion in the Yang–Mills–Higgs model in $\mathbb{R}^{1,3}$ [22, 5, 23]. The algebraic topology of these moduli spaces captures the spectrum of bound states at the quantum level, and it has been shown to be consistent with certain S-duality conjectures in quantum field theory [28, 27].

Monopoles in hyperbolic space, where the metric in polar coordinates is taken to be

$$ds^2 = dr^2 + R^4 \sinh^2 \left(\frac{r}{R^2} \right) (d\theta^2 + \sin^2 \theta d\phi^2),$$

were first studied by Atiyah [3], and they were soon perceived as being quite distinct from euclidean monopoles. They have a remarkably rich structure, which somewhat degenerates when the euclidean limit $R \rightarrow \infty$ is taken. One interesting fact about hyperbolic monopoles is that they are completely determined by the value of the connection at infinity [26, 24], which reduces to a $U(1)$ -connection on a 2-sphere; this can be regarded as a classical manifestation of the AdS/CFT correspondence [21]. The connection at infinity also plays a rôle in a characterisation of hyperbolic monopoles through holomorphic spheres in projective spaces [24], which is not available in the euclidean case.

A fundamental feature of hyperbolic monopoles that makes them different from their euclidean counterparts is that they come with different values of the mass $m \in]0, \infty[$ defined by equation (2). This parameter is related to the radius of curvature R of hyperbolic space: a rescaling of the radius by $R \mapsto \lambda R$ maps monopoles of mass m to monopoles of mass m/λ . (In the euclidean case, the radius of curvature is infinite and this means that euclidean monopoles of different masses can be identified.) Fixing the radius of hyperbolic space, the mass is a physical parameter and naturally appears as one of the moduli for the solutions of the Bogomol'nyi equations. The objects obtained in the limit of mass zero (sometimes called “nullarons”¹) are somewhat special and simpler to study; their twistor data have been pointed out to be directly related to complex curves arising in

¹This term is due to Michael Murray.

a construction of solutions to the Yang–Baxter equation for the chiral Potts model [6, 4]. By the same scaling argument, hyperbolic monopoles with infinite mass can be identified with euclidean monopoles [18].

One of the most basic questions one can ask about the moduli space of hyperbolic monopoles with a given topology is how this space is foliated by the mass. The answer to this question is trivial for monopoles of magnetic charge one. For charge two, a partial answer has been given in reference [24], where a distribution tangent to the leaves of this foliation was constructed. In this paper, we will give a much better characterisation of the mass of a two-monopole. We focus on the description of hyperbolic monopoles in terms of spectral curves, and parametrise spectral curves of 2-monopoles explicitly in terms of the mass parameter. In this way, we obtain a complete characterisation of all the moduli of two-monopoles in hyperbolic space. Our methods will apply to calculate the monopole mass in a continuous family of spectral curves of any charge, but are most effective in situations analogous to the charge two case, where we can take advantage of the fact that each spectral curve is elliptic. In higher charge, one can try to obtain spectral curves that are Galois covers of an elliptic curve by imposing platonic symmetries on the monopole fields, although such symmetric curves will not always correspond automatically to smooth solutions of the Bogomol’nyi equation (1). We shall undertake a systematic study of this class of curves, and use their symmetry to reduce the computation of the mass dependence, once again, to a calculation on elliptic curves. Thus we shall provide the first explicit examples of spectral curves of hyperbolic monopoles of arbitrary mass $m > 0$ and charges two, three and four beyond the (rather degenerate) axially symmetric case treated in reference [24]. We also investigate some limiting cases of our constructions; in particular, we will show how one can study the infinite-mass limit to recover spectral curves of euclidean monopoles that have featured in the existing literature [17, 14, 16].

2 Spectral curves of hyperbolic monopoles

The setup for twistor theory of hyperbolic space H^3 is the correspondence

$$\begin{array}{ccc} & \mathcal{C} & \\ \mu \swarrow & & \searrow \nu \\ H^3 & & Z \end{array} \quad (3)$$

that we now explain. We consider the model of H^3 as the upper half-plane $x_3 > 0$ in \mathbb{R}^3 , which we parametrise using cartesian coordinates (x_1, x_2, x_3) . It is useful to compactify H^3 by adding a boundary $\partial H^3 \cong \mathbb{P}^1$ (the 2-sphere S_∞^2 at infinity) obtained as a one-point compactification of the 2-plane of equation $x_3 = 0$. We use $z = x_1 + ix_2$ as a stereographic coordinate on this Riemann sphere ($z = \frac{z_1}{z_0}$ in terms of homogeneous coordinates for this \mathbb{P}^1 , and $z = \infty$ denotes the point at infinity). The geodesics on H^3 are the half-circles in \mathbb{R}^3 lying on planes perpendicular to and centred at points of $x_3 = 0$, together with the half-lines perpendicular to this plane, and are uniquely determined by a pair of points of intersection with ∂H^3 . So the space of all oriented geodesics (the twistor space of H^3) is

the complex surface

$$Z = \mathbb{P}^1 \times \mathbb{P}^1 - \mathbb{P}_{\Delta}^1,$$

where

$$\mathbb{P}_{\Delta}^1 := \{(w, z) \in \mathbb{P}^1 \times \mathbb{P}^1 : \hat{w} \neq z\}$$

is the antidiagonal; (w, z) denotes the oriented geodesic starting at the antipodal point $\hat{w} := -1/\bar{w}$ of w and ending at z . The correspondence space \mathcal{C} in (3) is the subset of $H^3 \times Z$ defined by the incidence relation

$$(x_1, x_2, x_3) \text{ lies on } (w, z), \quad (4)$$

while the maps μ and ν are the natural projections. Thus $\mu \circ \nu^{-1}(w, z)$ is the geodesic of H^3 corresponding to the oriented geodesic (w, z) , while $\nu \circ \mu^{-1}(x_1, x_2, x_3)$ is called the star at (x_1, x_2, x_3) and is the set of all oriented geodesics through this point of H^3 . There are two natural maps $\varepsilon_{\pm} : Z \rightarrow \mathbb{P}^1$

$$\varepsilon_{-}(w, z) = \hat{w}, \quad \varepsilon_{+}(w, z) = z$$

giving the endpoints of an oriented geodesic.

Lemma 2.1. The star at (x_1, x_2, x_3) is the projective line in Z given by the equation

$$(x_1 - ix_2)wz - (x_1^2 + x_2^2 + x_3^2)w + z - (x_1 + ix_2) = 0. \quad (5)$$

Proof. We observe that if (w, z) is in the star at $(0, 0, x_3)$, then z and w must have the same argument, and Pythagoras' theorem gives

$$\left| \frac{\hat{w} + z}{2} \right|^2 = x_3^2 + \left| \frac{\hat{w} - z}{2} \right|^2 \quad \Leftrightarrow \quad z = x_3^2 w, \quad (6)$$

which is the equation for the star at $(0, 0, x_3)$. This star is related to the star at (x_1, x_2, x_3) by a translation which maps an oriented geodesic (w', z') satisfying (6) to an oriented geodesic (w, z) through (x_1, x_2, x_3) , where $z' = z - (x_1 + ix_2)$ and

$$w' = \frac{w}{1 + (x_1 - ix_2)w}.$$

Substituting this in equation (6), we obtain (5). It is clear that (6) defines a projective line, and so the same must be true for the general star (5). \square

Note that the statements (4) and (5) are equivalent. A star (5) is invariant under the map

$$\sigma : (w, z) \mapsto (\hat{z}, \hat{w}) \quad (7)$$

reversing the orientation of oriented geodesics. This map is antiholomorphic and squares to the identity, so it is a reality structure for the complex surface Z ; it has no fixed points. A set invariant under σ is said to be real.

Twistor theory consists of interpreting analytic objects in physical space H^3 in terms of algebraic objects in twistor space Z , and vice-versa. In our context, the correspondence will relate hyperbolic monopoles (d_A, Φ) on H^3 to their spectral curves, which are complex curves $\Sigma \subset Z$ satisfying certain conditions. We now review briefly how they arise and refer the reader to references [25] and [24] for further details.

The twistor correspondence for the Bogomol'nyi equation (1) makes use of the family of operators

$$\mathfrak{H} : Z \longrightarrow \bigoplus_{(w,z) \in Z} \text{End } H^0(\mu \circ \nu^{-1}(w, z), \mathcal{E}).$$

introduced by Hitchin (for euclidean monopoles) in [12], and defined by

$$\mathfrak{H}(w, z) := (d_A - i\Phi) |_{\mu \circ \nu^{-1}(w, z)}.$$

The Bogomol'nyi equation (1) turns out to be the integrability condition for the holomorphic structure $(\mu \circ \nu^{-1})^* \bar{\partial}_A$ on the bundle $\ker \mathfrak{H} \rightarrow Z$ induced by the connection d_A . Thus, for a solution (d_A, Φ) of (1), $\ker \mathfrak{H}$ is a holomorphic complex bundle of rank two, and it has two distinguished line subbundles L^\pm with fibres

$$L_{(w,z)}^\pm = \left\{ s \in \ker \mathfrak{H}(w, z) : \lim_{x \rightarrow \varepsilon_\pm(w, z)} s(x) = 0 \right\}.$$

The spectral curve $\Sigma \subset Z$ of the monopole (d_A, Φ) is defined as the support of the cokernel of a morphism of coherent sheaves on Z given by the composition

$$\mathcal{O}(L^-) \longrightarrow \mathcal{O}(\ker \mathfrak{H}) \longrightarrow \mathcal{O}(\ker \mathfrak{H}/L^+),$$

which is the same as saying

$$\Sigma := \left\{ (w, z) \in Z : L_{(w,z)}^+ = L_{(w,z)}^- \right\}.$$

Holomorphic line bundles on Z can be constructed by purely algebraic means. They are obtained as tensor products of bundles pulled back from each factor of $\mathbb{P}^1 \times \mathbb{P}^1$,

$$\mathcal{O}_Z(k_1, k_2) := \hat{\varepsilon}_-^* \mathcal{O}_{\mathbb{P}^1}(k_1) \otimes \varepsilon_+^* \mathcal{O}_{\mathbb{P}^1}(k_2), \quad k_1, k_2 \in \mathbb{Z},$$

and complex powers L^λ ($\lambda \in \mathbb{C}$) of the topologically trivial line bundle

$$L := \mathcal{O}_Z(1, -1).$$

The bundles L^λ do not extend to $\mathbb{P}^1 \times \mathbb{P}^1 \supset Z$ unless $\lambda \in \mathbb{Z}$. We shall write

$$L^\lambda(k_1, k_2) := L^\lambda \otimes \mathcal{O}_Z(k_1, k_2). \tag{8}$$

These line bundles can be constructed as follows. Introduce

$$\begin{aligned} U_1 &:= \{(w, z) \in Z : w \neq \infty, z \neq \infty\}, \\ U_2 &:= \{(w, z) \in Z : w \neq 0, z \neq 0\}. \end{aligned} \tag{9}$$

since $(0, \infty), (\infty, 0) \in \mathbb{P}_\Delta^1$, these contractible sets provide an open cover of the twistor space Z . It can be taken as a trivialising cover for the line bundle (8), with transition function $g_{12} : U_1 \cap U_2 \rightarrow \mathbb{C}^*$ given by

$$g_{12}(w, z) = w^{k_1+\lambda} z^{k_2-\lambda}. \quad (10)$$

The connection between the holomorphic theory of the Bogomol'nyi equation above and algebraic geometry is provided by the identifications

$$L^+ = L^m(0, -k) \quad \text{and} \quad L^- = L^{-m}(-k, 0)$$

for a hyperbolic monopole of charge k and mass m . This result is obtained from an analysis of the asymptotics of the fields (d_A, Φ) near the boundary ∂H^3 [25]. Spectral curves of hyperbolic monopoles satisfy the following three conditions:

1. $\Sigma \subset Z$ is a real curve in the linear system $|\mathcal{O}_Z(k, k)|$;
2. $L^{2m+k}|_\Sigma \cong \mathcal{O}_\Sigma$;
3. $H^0(\Sigma, L^s(k-2, 0)) = 0$ for all $0 < s < 2m+2$.

In the broader context of integrable systems, these conditions lead to an ODE of Lax type. It is believed that they are also sufficient to guarantee the existence of a hyperbolic monopole to which Σ is associated. Although sufficiency has not yet been proven, for the purposes of this paper we will use the terms hyperbolic monopole and spectral curve to refer to complex curves in Z satisfying the conditions above.

Using the coordinates w, z for Z , one can write a polynomial equation for Σ as

$$\psi(w, z) = (-w)^k \langle q(\hat{w}), q(z) \rangle$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in \mathbb{C}^{k+1} and $q : \mathbb{C} \rightarrow \mathbb{C}^{k+1}$ is defined in terms of $k+1$ vectors $v_j \in \mathbb{C}^{k+1}$ by

$$q(z) = \sum_{j=0}^k \sqrt{\binom{k}{j}} z^j v_j;$$

in fact, q induces a holomorphic map $\mathbb{P}^1 \rightarrow \mathbb{P}^k$ that also characterises the monopole up to the action of $\text{PU}(k+1)$ on the target [24].

The conditions 1.–3. above yield a model for the moduli space of hyperbolic k -monopoles \mathcal{M}_k in terms of spectral curves. A version of this was used in reference [24] to give a description of \mathcal{M}_2 . The group of (direct) isometries of H^3 can be identified with $\text{PSL}_2\mathbb{C}$, and it acts on \mathcal{M}_k . There is a moment map associated to this action, which can be used to define a centre of a monopole: spectral curves in the zero-set of the moment map correspond to centred monopoles and satisfy [24]

$$\sum_{j=0}^k (2j-k) \|v_j\|^2 = 0, \quad \sum_{j=0}^{k-1} \sqrt{(j+1)(k-j)} \langle v_j, v_{j+1} \rangle = 0.$$

3 The mass of hyperbolic 2-monopoles

Our aim in this section is to obtain an explicit equation for the spectral curve $\Sigma \subset Z$ of a hyperbolic 2-monopole in terms of its mass. We shall first deal with the case where Σ is smooth, and then show that the result can be extended by continuity to arbitrary spectral curves.

3.1 A model for the spectral curve

We begin with a lemma giving a standard form for smooth spectral curves of hyperbolic 2-monopoles.

Lemma 3.1. If the spectral curve of a 2-monopole is smooth, its $\mathrm{PSL}_2\mathbb{C}$ -orbit contains a curve $\Sigma \subset Z$ of the form

$$\psi(w, z) = w^2 z^2 + \frac{u^2 - 2uv + 4}{2(u - v)} wz - \frac{u^2 - 4}{4(u - v)}(w^2 + z^2) + 1 = 0. \quad (11)$$

Here, $u, v \in]2, \infty[$ depend only on the monopole mass and on an internal modulus, and they satisfy

$$u^2 - 2uv + 4 > 0. \quad (12)$$

Proof. The spectral curve Σ of a 2-monopole is the vanishing locus of a polynomial of the form

$$\begin{aligned} \psi(w, z) = & \langle v_0, v_2 \rangle w^2 z^2 - \sqrt{2} \langle v_1, v_2 \rangle (w^2 z + w z^2) + \|v_0\|^2 (w^2 + z^2) - \\ & - 2\|v_1\|^2 w z + \sqrt{2} \langle v_2, v_1 \rangle (w + z) + \langle v_2, v_0 \rangle, \end{aligned} \quad (13)$$

where $v_j \in \mathbb{C}^3$. The intersection of Σ with the diagonal $\mathbb{P}_\Delta^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$, which we parametrise using the coordinate z , is then given by the zeroes of the quartic polynomial $\psi(z, z)$. From (13) we find

$$\psi(\hat{z}, \hat{z}) = \bar{z}^{-4} \overline{\psi(z, z)},$$

which implies that these zeroes occur in antipodal pairs. Moreover, these points must be distinct for Σ to be smooth. So we can use an element of $\mathrm{Stab}_{(0,0,1)} \mathrm{PSL}_2\mathbb{C} \cong \mathrm{SO}(3)$ to rotate them to the positions $\pm\sqrt{\lambda}$ and $\pm\frac{1}{\sqrt{\lambda}}$ on the real axis of \mathbb{P}_Δ^1 . The value of $\lambda \in]0, 1[$ is uniquely determined by this process, once the mass of the monopole has been fixed. Thus we have shown that a rotation can be applied to have

$$\psi(z, z) = z^4 - \left(\lambda + \frac{1}{\lambda} \right) z^2 + 1; \quad (14)$$

in this step, we have used the freedom of multiplying ψ by an overall factor in \mathbb{C}^* . The most general form for $\psi(w, z)$ consistent with (14) is

$$\psi(w, z) = w^2 z^2 + \Lambda^2 w z - \frac{1}{2} \left(\lambda + \frac{1}{\lambda} + \Lambda^2 \right) (w^2 + z^2) + 1 = 0.$$

Invariance under the reality map only constrains Λ^2 to be a real number. However, given this, the positivity condition on spectral curves can easily be seen to be equivalent to $\Lambda^2 > 0$. To complete the proof, we need only to observe that (by an easy continuity argument) the map $(u, v) \mapsto (\lambda + \frac{1}{\lambda}, \Lambda^2)$ defined by

$$\begin{aligned} \{(u, v) \in]2, \infty[^2 : u^2 - 2uv + 4 > 0\} &\longrightarrow]2, \infty[\times]0, \infty[\\ (u, v) &\longmapsto \left(\frac{uv-4}{u-v}, \frac{u^2-2uv+4}{2(u-v)} \right) \end{aligned} \quad (15)$$

is bijective, and that

$$u > v \quad (16)$$

is always satisfied in the domain of (15). \square

Our task is to calculate the dependence of u and v on the mass m and an extra real parameter. As a preliminary step, we need to obtain a good description of the family of elliptic curves $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1$ given by (11). We introduce the map

$$\begin{aligned} \pi : \mathbb{P}^1 \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^2 \\ ([w_0 : w_1], [z_0 : z_1]) &\longmapsto [w_0 z_0 : w_0 z_1 + w_1 z_0 : w_1 z_1]. \end{aligned} \quad (17)$$

which can be regarded as a projection onto the space of orbits of the automorphism

$$\sigma_+ : (w, z) \mapsto (z, w). \quad (18)$$

This map has order two, commutes with the reality structure σ defined in (7), and can be written as

$$\sigma_+ = \sigma \circ \sigma_- = \sigma_- \circ \sigma,$$

where

$$\sigma_- : (w, z) \mapsto (\hat{w}, \hat{z}); \quad (19)$$

σ_- is induced on $\mathbb{P}^1 \times \mathbb{P}^1$ by the parity transformation on hyperbolic space

$$\mathcal{P}_{(0,0,1)} : (x_1, x_2, x_3) \mapsto \left(-\frac{x_1}{x_1^2 + x_2^2}, \frac{x_2}{x_1^2 + x_2^2}, \frac{1}{x_3} \right), \quad (20)$$

which can be described as reflection across $(0, 0, 1)$. Since $\mathcal{P}_{(0,0,1)}$ reverses the orientation of H^3 , σ_- is antiholomorphic, and it is a reality structure on $\mathbb{P}^1 \times \mathbb{P}^1$ or Z alternative to (7). Thus there is a Vierergruppe of diffeomorphisms of twistor space associated with $(0, 0, 1)$ (and likewise with any other point of H^3),

$$\mathcal{V}_{(0,0,1)} := \{\text{id}, \sigma, \sigma_-, \sigma_+\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2,$$

which is naturally \mathbb{Z}_2 -graded by (anti)holomorphicity. That σ_+ should arise as a symmetry of spectral curves (13) of centred 2-monopoles is unsurprising, as we expect their fields to be symmetric under the parity transformation (20).

The fixed points of σ_+ form the diagonal \mathbb{P}_Δ^1 of $\mathbb{P}^1 \times \mathbb{P}^1$ (i.e. the star at $(0, 0, 1)$), and $\sigma_+^2 = \text{id}$. Thus π is two-to-one on $\mathbb{P}^1 \times \mathbb{P}^1 - \mathbb{P}_\Delta^1$ and one-to-one on \mathbb{P}_Δ^1 . In homogeneous coordinates Z_0, Z_1 and Z_2 for \mathbb{P}^2 , $\pi(\mathbb{P}_\Delta^1)$ is the conic given by

$$C(Z_0, Z_1, Z_2) := Z_1^2 - 4Z_0Z_2 = 0. \quad (21)$$

The image of Σ under π is given by the equation

$$\tilde{\psi}(\zeta_1, \zeta_2) := \zeta_2^2 - \frac{u^2 - 4}{4(u - v)} \zeta_1^2 + u\zeta_2 + 1 = 0$$

in affine coordinates $\zeta_1 = \frac{Z_1}{Z_0} = w + z$ and $\zeta_2 = \frac{Z_2}{Z_0} = wz$. This is also a plane conic, so we can find a rational parametrisation $f : \mathbb{P}^1 \rightarrow \pi(\Sigma)$ for it. In fact, if we write

$$f(t) = [1 - \alpha t^2 : 2\alpha\beta t : t^2 - \alpha], \quad (22)$$

we will have

$$f(\mathbb{P}^1) = \pi(\Sigma)$$

if and only if we choose α and β such that the conditions

$$u = \alpha + \frac{1}{\alpha} \quad (23)$$

and

$$u - v = \alpha\beta^2 \quad (24)$$

are satisfied; α and β are then real. Notice that the equation (23) determining α indeed gives two real solutions of the form $\alpha, \frac{1}{\alpha}$ since $u > 2$, and because of the invariance under $\alpha \mapsto \frac{1}{\alpha}$ we are free to assume that

$$\alpha > 1,$$

which we shall do from now on; that β is real then follows from (16). We now realise Σ as a double cover of the \mathbb{P}^1 where f is defined, branching on the set of four points $f^{-1}(\pi(\Sigma) \cap \pi(\mathbb{P}_\Delta^1))$. One way to do this is to regard Σ as the Riemann surface of the global function of t extending the germ

$$t \mapsto \sqrt{F(t)} \quad (25)$$

at $t = 0$, where

$$F(t) := C \circ f(t) = 4\alpha(t^4 - vt^2 + 1). \quad (26)$$

(In (25) and henceforth, we use $\sqrt{\cdot}$ to denote the principal branch of the square root.) The branch points of $\Sigma \rightarrow \mathbb{P}^1$ are the zeroes of F , and they are of the form $\pm\sqrt{\kappa}, \pm\frac{1}{\sqrt{\kappa}}$ with

$$\kappa + \frac{1}{\kappa} = v. \quad (27)$$

Again, $v > 2$ ensures that κ is real and positive, and by invariance of (27) under $\kappa \mapsto \frac{1}{\kappa}$ we may assume from now on that

$$0 < \kappa < 1. \quad (28)$$

The elliptic curve Σ given by (11) can now be realised as a two-sheeted cover of the projective line parametrised by t as follows (cf. Figure 1). Introduce branch cuts on the segments $[-\frac{1}{\sqrt{\kappa}}, -\sqrt{\kappa}]$ and $[\kappa, \frac{1}{\sqrt{\kappa}}]$, and label the two sheets as (i) and (ii), where sheet (i) is the one on which the germ (25) (which takes a positive value at $t = 0$) is defined. Clearly, on sheet (i)/(ii), the analytic continuation of (25) then takes negative/positive values on $] -\infty, -\frac{1}{\sqrt{\kappa}}[$ and $] \frac{1}{\sqrt{\kappa}}, +\infty[$, and positive/negative values on $] -\sqrt{\kappa}, \sqrt{\kappa}[$, respectively.

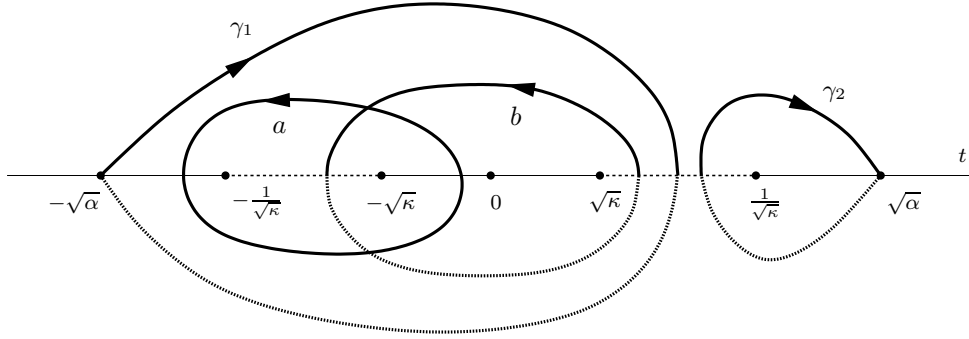


Figure 1: Hyperbolic 2-monopole: contours of integration in a realisation of Σ

To complete the correspondence between our descriptions of Σ as the Riemann surface of the germ (25) and as a hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1$, we just need to make the identification of the two descriptions at a single point. We choose this point p_1 to be an element of the fibre of $\Sigma \rightarrow \mathbb{P}^1$ over $t = -\sqrt{\alpha}$. In the description of Σ by analytic continuation, we specify p_1 by saying that it lies on sheet (i). In the description $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1$, we will need to specify the coordinates (w, z) of p_1 . Notice that from (22) we have that when $t = \mp\sqrt{\alpha}$

$$wz = \frac{(\mp\sqrt{\alpha})^2 - \alpha}{1 - \alpha(\mp\sqrt{\alpha})^2} = 0,$$

so either $w = 0$ or $z = 0$; we specify p_1 by saying that it has $z = 0$. Then we find from (11) that $w = \pm 2\sqrt{\frac{u-v}{u^2-4}}$; we must set

$$p_1 := \left(2\sqrt{\frac{u-v}{u^2-4}}, 0 \right), \quad (29)$$

since (17) implies that the point

$$p_2 := \left(-2\sqrt{\frac{u-v}{u^2-4}}, 0 \right) \quad (30)$$

projects to $t = +\sqrt{\alpha}$ under (17). For further reference, we note that the other points on

the fibres over $t = \mp\sqrt{\alpha}$ have $w = 0$ and $z = \pm 2\sqrt{\frac{u-v}{u^2-4}}$, and we label them as

$$q_1 := \left(0, 2\sqrt{\frac{u-v}{u^2-4}}\right), \quad q_2 := \left(0, -2\sqrt{\frac{u-v}{u^2-4}}\right). \quad (31)$$

By construction, p_1 is on sheet (i), so q_1 must be on sheet (ii) as it belongs to the same fibre. Moreover, one can write

$$w(t) = \frac{Z_1(t) \pm \sqrt{F(t)}}{2Z_0(t)}$$

where the signs \pm distinguish between the two sheets; this equation implies that, on a given sheet, if $w = 0$ at $t = \mp\sqrt{\alpha}$, then $w \neq 0$ at $t = \pm\sqrt{\alpha}$, and the same is true for z . So we conclude that p_2 is on sheet (ii), while q_2 is on sheet (i).

We want to discuss the conditions characterising a spectral curve Σ from the point of view of the function theory on Σ , and for that we shall need to fix generators for the spaces of 1-forms and 1-cycles on the elliptic curve. A global holomorphic 1-form on Σ is given by the standard formula

$$\omega = \frac{dt}{\pm\sqrt{F(t)}} \quad (32)$$

which follows from taking Poincaré residues of (11) [11]; again, the signs label sheets, and we make the convention of taking the top sign on sheet (i) near $t = 0$. To describe a standard basis (a, b) for $H_1(\Sigma; \mathbb{Z}) \cong \mathbb{Z}^2$ we draw representatives for a and b in Figure 1. The convention is that paths on sheet (i) are drawn with continuous lines, whereas paths on sheet (ii) are drawn with dotted lines.

3.2 Reciprocity on Σ

To make the connection between the algebraic-geometric condition 2. and complex analysis on Σ , we shall use the reciprocity law for differentials of first and third kinds on a compact Riemann surface (cf. [11], pp. 229–230). This technique is in the same spirit of previous work on euclidean monopoles, where a different reciprocity law has been applied to deduce constraints on the coefficients of polynomials defining spectral curves [17, 8, 15].

We start by rephrasing $L^{2m+k}|_{\Sigma} \cong \mathcal{O}_{\Sigma}$ in terms of the existence of a global holomorphic trivialisation for the line bundle $L^{2(m+1)}$ on Σ , or equivalently a nowhere-vanishing holomorphic section $\xi \in H^0(\Sigma, L^{2(m+1)})$. Such a section is locally represented by nowhere-vanishing holomorphic functions ξ_j defined on the basic open sets $U_j \cap \Sigma$ ($j = 1, 2$) (cf. (9)), and which are patched together using the appropriate transition function (10):

$$\xi_1(w, z) = \left(\frac{w}{z}\right)^{2(m+1)} \xi_2(w^{-1}, z^{-1}). \quad (33)$$

To begin with, this patching condition should hold on $U_1 \cap U_2 \cap \Sigma$, which consists of Σ with eight points deleted, namely the points of the form $(w, 0)$, (w, ∞) , $(0, z)$ and (∞, z) . There are two of each, since the polynomial $\psi(w, z)$ defining Σ has bidegree $(2, 2)$; for

example, the points of the form $(0, z)$ are p_1 and p_2 given by equations (29) and (30). Taking logarithmic differentials in both sides of (33), we obtain the relation

$$\omega_1 = 2(m+1) \left(\frac{dw}{w} - \frac{dz}{z} \right) + \omega_2 \quad (34)$$

where

$$\omega_j := d \log \xi_j = \frac{d\xi_j}{\xi_j} \quad j = 1, 2. \quad (35)$$

The definition (35) implies the following integrality property on periods:

$$\oint_a \omega_j, \oint_b \omega_j \in 2\pi i \mathbb{Z}. \quad (36)$$

We set

$$\ell_1 := \frac{1}{2\pi i} \oint_b \omega_1, \quad \ell_2 := -\frac{1}{2\pi i} \oint_a \omega_1. \quad (37)$$

Notice that we can replace $2(m+1)$ by s and go through the same argument to conclude that condition 3. can be rephrased as:

$$(\ell_1, \ell_2) \text{ must be primitive in } \mathbb{Z}^2. \quad (38)$$

Each ω_j is a holomorphic 1-form on $U_j \cap \Sigma$ (since ξ_j are holomorphic and never vanish there), but they extend to global meromorphic 1-forms on Σ as a consequence of (34). To see this, notice that ω_2 is holomorphic on a small neighbourhood of

$$\Sigma - U_1 = \{p_1, p_2, q_1, q_2\}$$

in Σ , and then (34) implies that ω_1 must have the same principal part as $2(m+1) \left(\frac{dw}{w} - \frac{dz}{z} \right)$ on this set. Since w is a good coordinate on Σ around p_1 and p_2 , while z is a good coordinate around q_1 and q_2 , we can conclude that ω_1 has simple poles at each of these points, and we find

$$\text{Res}_{p_j}(\omega_1) = -2(m+1) \quad \text{and} \quad \text{Res}_{q_j}(\omega_1) = 2(m+1), \quad j = 1, 2. \quad (39)$$

A similar analysis can be done for ω_2 .

In classical terminology, a holomorphic 1-form (such as ω) is called a differential of the first kind, while a meromorphic 1-form with only single poles (such as ω_1) is called a differential of the third kind. A standard result of complex analysis on compact Riemann surfaces is the reciprocity law [11]

$$\left| \begin{array}{cc} \oint_a \omega & \oint_a \omega_1 \\ \oint_b \omega & \oint_b \omega_1 \end{array} \right| = 2\pi i \sum_q \text{Res}_q(\omega_1) \int_{p_0}^q \omega. \quad (40)$$

Here, $p_0 \in \Sigma$ is any basepoint, the integration paths on the right-hand side can be deformed to avoid the paths representing the homology basis, and the sum is over the set of poles of ω_1 . Equation (40) relates two elements of the covering space $H^1(\Sigma, \mathcal{O}_\Sigma) \cong T_{\mathcal{O}_\Sigma} \text{Jac}(\Sigma)$

to the jacobian variety of Σ . On a complex curve of higher genus g , the left-hand-side of the reciprocity relation would have a sum over conjugate pairs of a standard (symplectic) basis of $H_1(\Sigma; \mathbb{Z})$, and there would be g equations. Note that by Liouville's theorem

$$\sum_q \text{Res}_q(\omega_1) = 0,$$

which explains why the right-hand side of (40) is in fact independent of p_0 .

In our context, using (37) and (39), equation (40) can be written as

$$\ell_1 \oint_a \omega + \ell_2 \oint_b \omega = 2(m+1) \left(\int_{p_1}^{q_1} \omega + \int_{p_2}^{q_2} \omega \right). \quad (41)$$

We note in passing that elements of $H^0(\Sigma, K_\Sigma)^* \cong H^1(\Sigma, \mathcal{O}_\Sigma)$ are realised in different ways in the two sides of this equation, namely as a pairing (via \oint) with a 1-cycle $\ell_1 a + \ell_2 b \in H_1(\Sigma; \mathbb{Z})$ in the left-hand side, and as a multiple of an Abel sum associated to the divisor $q_1 + q_2 - p_1 - p_2 \in \text{Div}_0(\Sigma)$ representing the class $\mathcal{O}(1, -1)$ in the right-hand side; in fact, if we interpret the integrals in terms of the Abel–Jacobi map, and use congruence on the period lattice of Σ , we can read (41) as a rather transparent statement of $L^{2(m+1)}|_\Sigma$ being trivial. Of course, there is a freedom of changing the subscripts of the p_j and q_j in this expression, but we find it convenient to choose paths of integration as in (41). More precisely, we choose paths γ_j ($j = 1, 2$) on Σ starting at p_j and ending at q_j as illustrated in Figure 1, but defined only as a class in the relative homology of $(\Sigma, \{p_j, q_j\})$. Notice that the locations of the projections $t = \pm\sqrt{\alpha}$ of the poles of ω_1 with respect to the branch points are as shown, since

$$u > v \Leftrightarrow \alpha + \frac{1}{\alpha} > \kappa + \frac{1}{\kappa} \Rightarrow \alpha > \frac{1}{\kappa}.$$

To understand what constraint (41) imposes on the coefficients of Σ , we have to compute the integrals in this equation. Under our conventions, it is clear that all of them are real except for the period $\oint_a \omega$, which is pure imaginary. Therefore $\ell_1 = 0$, and (38) imposes $\ell_2 = \pm 1$. So we only need to evaluate

$$\oint_b \omega = -2I_1,$$

$$\int_{\gamma_1} \omega = 2(I_1 - I_2)$$

and

$$\int_{\gamma_2} \omega = -2I_2,$$

where

$$I_1 := \int_{-\sqrt{\kappa}}^{\sqrt{\kappa}} \frac{dt}{\sqrt{F(t)}}, \quad I_2 := \int_{\frac{1}{\sqrt{\kappa}}}^{\sqrt{\alpha}} \frac{dt}{\sqrt{F(t)}}.$$

In terms of standard notation for elliptic integrals of the first kind

$$F(\varphi, \kappa) := \int_0^\varphi \frac{d\theta}{\sqrt{1 - \kappa^2 \sin^2 \theta}},$$

$$K(\kappa) := F\left(\frac{\pi}{2}, \kappa\right),$$

we calculate

$$I_1 = \sqrt{\frac{\kappa}{\alpha}} K(\kappa)$$

and

$$I_2 = \frac{1}{2} \sqrt{\frac{\kappa}{\alpha}} \left(K(\kappa) - F\left(\arcsin \frac{1}{\sqrt{\alpha\kappa}}, \kappa\right) \right).$$

Substituting in equation (41), we find that $\ell_1 = -1$ and

$$F\left(\arcsin \frac{1}{\sqrt{\alpha\kappa}}, \kappa\right) = \frac{K(\kappa)}{2(m+1)}. \quad (42)$$

3.3 Mass parametrisation

Equation (42) can be solved for α to give

$$\alpha = \frac{1}{\kappa \operatorname{sn}^2\left(\frac{K(\kappa)}{2(m+1)}, \kappa\right)} \quad (43)$$

in terms of Jacobi's sine amplitude function. (In the following, we will often drop the modulus argument κ in jacobian elliptic functions.) Using (27), (43) and standard algebra of jacobian functions [7], we find

$$\frac{u^2 - 2uv + 4}{2(u-v)} = \frac{2}{\kappa} \operatorname{cs}\left(\frac{K(\kappa)}{m+1}\right) \operatorname{ds}\left(\frac{K(\kappa)}{m+1}\right) \quad (44)$$

and

$$\frac{u^2 - 4}{4(u-v)} = \frac{1}{\kappa} \operatorname{ns}^2\left(\frac{K(\kappa)}{m+1}\right). \quad (45)$$

So we have shown that equation (11) defines a 2-monopole of mass m if and only if (44) and (45) hold. Moreover, we shall see in section 4.1 below that we can extend our results by continuity to the $\kappa \rightarrow 0$ limit, where the spectral curve becomes reducible and singular. In other words, we have the following result.

Proposition 3.2. A spectral curve of a 2-monopole of mass m centred at $(0, 0, 1) \in H^3$ can be rotated to the form

$$\kappa \operatorname{sn}^2 \rho (w^2 z^2 + 1) + 2 \operatorname{cn} \rho \operatorname{dn} \rho w z - (w^2 + z^2) = 0, \quad (46)$$

where $0 \leq \kappa < 1$ and

$$\rho = \frac{K(\kappa)}{m+1}. \quad (47)$$

When m is a rational number, the coefficients of (46) turn out to be algebraic over $\mathbb{Q}(\kappa)$. For example, if $m = 1$ we can use bisection formulae for jacobian elliptic functions [7] to write the spectral curve in the form

$$\kappa(w^2 z^2 + 1) + 2\kappa' \sqrt{1 + \kappa'} w z - (1 + \kappa')(w^2 + z^2) = 0,$$

where $\kappa' := \sqrt{1 - \kappa^2}$. We shall encounter later a similar phenomenon occurs in examples of spectral curves of higher charge.

4 Limiting cases of 2-monopoles

In this section, we examine and interpret the limits of our result in Proposition 3.2 as $\kappa \rightarrow 0$, $\kappa \rightarrow 1$, $m \rightarrow 0$ and $m \rightarrow \infty$.

4.1 Axial symmetry

When we let $\kappa \rightarrow 0$,

$$\begin{aligned} \operatorname{sn} \rho &\rightarrow \sin \rho, \\ \operatorname{cn} \rho &\rightarrow \cos \rho, \\ \operatorname{dn} \rho &\rightarrow 1. \end{aligned}$$

whereas (47) implies

$$\rho \rightarrow \frac{\pi}{2(m+1)}.$$

Thus (46) becomes

$$w^2 - 2 \cos \left(\frac{\pi}{2(m+1)} \right) w z + z^2 = 0. \quad (48)$$

This is the spectral curve of an axially symmetric 2-monopole at $(0, 0, 1)$, as computed in reference [24]. This curve is reducible, since the polynomial in (48) factorises as

$$\left(w - e^{\frac{i\pi}{2(m+1)}} z \right) \left(w - e^{-\frac{i\pi}{2(m+1)}} z \right).$$

The reduced components are two projective lines in $\mathbb{P}^1 \times \mathbb{P}^1$, intersecting at the points $(0, 0)$ and (∞, ∞) in the diagonal, which are related by the real structure σ . Notice that the lines

$$w = e^{\pm \frac{i\pi}{2(m+1)}} z$$

are real and can be thought of as stars at the complex conjugate points $(0, 0, e^{\pm \frac{i\pi}{4(m+1)}})$ (cf. equation (6) for a star at $(0, 0, x_3) \in H^3$). This situation is analogous to axially symmetric monopoles in euclidean space [14].

We conclude that we can extend the range of κ from $0 < \kappa < 1$ to $0 \leq \kappa < 1$, as in Proposition 3.2.

4.2 Large separation

To study the limit $\kappa \rightarrow 1$, we use that [7]

$$\begin{aligned}\operatorname{sn} \rho &\rightarrow \tanh \rho, \\ \operatorname{cn} \rho &\rightarrow \operatorname{sech} \rho, \\ \operatorname{dn} \rho &\rightarrow \operatorname{sech} \rho,\end{aligned}$$

together with

$$\rho \rightarrow +\infty$$

from equation (47). We find that the spectral curve (46) then degenerates to

$$(w^2 - 1)(z^2 - 1) = 0. \quad (49)$$

This reduces to four real lines in $\mathbb{P}^1 \times \mathbb{P}^1$. The two lines $w = \mp 1$ and $z = \pm 1$ together can be regarded as a limiting star at the point $(\pm 1, 0, 0) \in \partial H^3$, for either choice of signs; these two lines intersect at the point $(\mp 1, \pm 1) \in \mathbb{P}_{\Delta}^1$, respectively. We interpret (49) as the spectral curve of two 1-monopoles that are infinitely separated at each of the ends of the geodesic $\mu \circ \nu^{-1}(1, 1) = \mu \circ \nu^{-1}(-1, -1)$ of H^3 .

The analysis of the limiting examples $\kappa \rightarrow 0$ and $\kappa \rightarrow 1$ suggests that κ can roughly be thought of as a parameter of the separation of two single monopole cores in configurations of 2-monopoles centred at $(0, 0, 1)$. As κ varies from 1 to 0, the cores approach symmetrically along the geodesic given by $x_2 = 0$, $x_1^2 + x_3^2 = 1$ from infinite distance to coincidence at $(0, 0, 1)$ (the axially symmetric configuration). Strictly speaking, the description of a 2-monopole configuration in terms of two superposed 1-monopoles is only appropriate in the asymptotic limit of large separation, and becomes worse and worse as κ decreases. This is why the 1-monopole cores in the axially symmetric configuration are found to be located at points of the complexification of H^3 rather than at the centre $(0, 0, 1)$ itself.

4.3 2-Nullarons

If we let $m \rightarrow 0$, then

$$\begin{aligned}\operatorname{sn} \rho &\rightarrow \operatorname{sn} K(\kappa) = 1, \\ \operatorname{cn} \rho &\rightarrow \operatorname{cn} K(\kappa) = 0, \\ \operatorname{dn} \rho &\rightarrow \operatorname{dn} K(\kappa) = \sqrt{1 - \kappa^2},\end{aligned}$$

and the spectral curve (46) becomes

$$\kappa(w^2 z^2 + 1) - (w^2 + z^2) = 0. \quad (50)$$

This result can be obtained by more direct means. In fact, (50) is the standard \mathbb{Z}_2 -symmetric curve encoding a solution of the Potts model [4], and can be computed as

$$\widehat{R(w)} = R(\hat{z}) \quad (51)$$

with

$$R(z) = \frac{\sqrt{1 - \kappa^2}}{z^2 - \kappa}.$$

More generally, equation (51) produces all the spectral curves of massless monopoles (nullarons) of charge k from rational maps $R : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree k .

4.4 Euclidean 2-monopoles

In the limit $m \rightarrow \infty$, which is equivalent to $\rho \rightarrow 0$, the curve (46) tends to two copies of the diagonal $\mathbb{P}_\Delta^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$, with equation $(w - z)^2 = 0$. However, at the next asymptotic order, we can recover the spectral curve of a euclidean monopole embedded in $T\mathbb{P}_\Delta^1$.

To explain this, we must first make more precise what is meant by the euclidean limit of a hyperbolic monopole. As discussed in the introduction, rescaling the hyperbolic metric to have larger and larger curvature radius R is equivalent to rescaling the fields (equivalently, the spectral curves), and in particular their mass, while keeping the metric constant; the infinite radius limit $R \rightarrow \infty$ can then be thought of as an infinite mass limit $m \rightarrow \infty$, to be interpreted as a euclidean monopole. Implicit in this rescaling is the choice of a point in H^3 . We choose to rescale around the centre $(0, 0, 1) \in H^3$, and this forces us to consider only centred hyperbolic monopoles with limit centred euclidean monopoles. More generally, rescaling around a different point in H^3 would lead to limit spectral curves concentrated around the corresponding star in $\mathbb{P}^1 \times \mathbb{P}^1$.

Recall that a point $(w, z) \in \mathbb{P}^1 \times \mathbb{P}^1$ is interpreted as a geodesic in H^3 running from \hat{w} to z , both points on the sphere at infinity. From the perspective of the centre $(0, 0, 1)$, the geodesic is viewed as the vector pointing to its closest point from $(0, 0, 1)$ and the (orthogonal) tangent direction at that point. Now take a sequence of geodesics in H^3 in the spectral curves converging to a geodesic through $(0, 0, 1)$, or equivalently a sequence of points $(w_m, z_m) \in \mathbb{P}^1 \times \mathbb{P}^1$ for $m \rightarrow \infty$ with limit $(w_\infty, z_\infty) = (z_\infty, z_\infty)$ on the diagonal. For each value of m , rescale the radius of curvature of the metric on H^3 by m . Then a simple geometric argument (involving similarity of infinitesimal triangles on the 2-plane containing the sequence of geodesics) shows that, in the limit, one obtains the euclidean geodesic

$$(w_m, z_m) \rightarrow (\eta, \zeta) = \left(\lim_{m \rightarrow \infty} m(z_m - w_m), z_\infty \right), \quad (52)$$

where we are using the standard coordinates on $T\mathbb{P}_\Delta^1$ introduced by Hitchin [12]. Given the $\rho \rightarrow 0$ asymptotics

$$\text{sn}^2(\rho, \kappa) = \rho^2 + O(\rho^3)$$

and

$$\text{cn}(\rho, \kappa) \text{dn}(\rho, \kappa) = 1 - \left(\frac{1 + \kappa^2}{2} \right) \rho^2 + O(\rho^3),$$

we can write

$$\begin{aligned}
\eta^2 &= \lim_{m \rightarrow \infty} m^2 (w_m - z_m)^2 \\
&= K(\kappa)^2 \lim_{\rho \rightarrow 0} \frac{(w_m - z_m)^2}{\rho^2} \\
&= K(\kappa)^2 \lim_{\rho \rightarrow 0} \frac{(2 \operatorname{cn} \rho \operatorname{dn} \rho - 2) w_m z_m + \kappa \operatorname{sn}^2 \rho (w_m^2 z_m^2 + 1)}{\rho^2} \\
&= K(\kappa)^2 \lim_{m \rightarrow \infty} (-(1 + \kappa^2) w_m z_m + \kappa (w_m^2 z_m^2 + 1)) \\
&= K(\kappa)^2 (-(1 - \kappa^2) \zeta^2 + \kappa (\zeta^4 - 1)),
\end{aligned}$$

where we have used (52) in the first and last steps. Hence we obtain the curve in $|\mathcal{O}_{T\mathbb{P}^1}(4)|$

$$\eta^2 - K(\kappa)^2 (\zeta^2 - \kappa) (\kappa \zeta^2 - 1) = 0 \quad (53)$$

as the euclidean limit of (46). To compare this with the spectral curve of a generic euclidean 2-monopole [17], as given in reference [23] (say),

$$\eta^2 - \frac{K(k)^2}{4} (k^2 (\zeta^4 + 1) - 2(2 - k^2) \zeta^2) = 0, \quad (54)$$

where $0 < k < 1$, one can start by relating κ and k by computing the coordinate ζ at the four intersection points of (53) and (54) with the zero section $\eta = 0$ of $T\mathbb{P}^1$ (i.e. the branch points of $\Sigma \rightarrow \mathbb{P}^1$). In (53) we find $\zeta = \pm \sqrt{\kappa}, \pm \frac{1}{\sqrt{\kappa}}$, whereas for (54) the solutions have the same form but are parametrised differently, namely

$$\kappa = \frac{2}{k^2} - 1 - \sqrt{\left(\frac{2}{k^2} - 1\right)^2 - 1} \quad \Rightarrow \quad k = \frac{2\sqrt{\kappa}}{1 + \kappa}.$$

Using a descending Landen's transformation [9], we can now write

$$K(k) = K\left(\frac{2\sqrt{\kappa}}{1 + \kappa}\right) = (1 + \kappa)K(\kappa)$$

and conclude that (53) and (54) simply give two different parametrisations of the same curve. An alternative way of checking that our limit curve is correct is to start from our parametrisation of the branch points and use the characterisation of spectral curves in section 3 of [15] to deduce (53).

5 Platonic monopoles in hyperbolic space

For charge $k > 2$, the spectral curves are of higher genus $(k - 1)^2 > 1$ and depend on $4k - 3$ internal moduli, so it becomes a difficult task to compute any one of them, let alone obtain a complete picture of the whole moduli space. A technique that has been fruitful is to impose invariance of the monopoles under certain isometries [14]; in some cases, this

cuts the number of moduli down to a manageable number, while important features of the moduli space are preserved. For instance, such symmetry constraints define totally geodesic subsets of the moduli space, whose geodesic flow is simply a restriction of the ambient geodesic flow; moreover, one expects these subsets to carry significant information about the topology of the whole moduli space. In the euclidean case, beautiful examples of scattering of symmetric multi-monopoles have been found in this way [14, 23], as well as new insight into the structure of the monopole fields themselves [29].

Usually, the construction of spectral curves of symmetric monopoles on euclidean space has relied on the study of Nahm's equations. For hyperbolic monopoles of general mass this route cannot be taken, as we do not yet have enough understanding of the generalisation of Nahm's equations appropriate for the problem. In the following sections, we shall obtain results about symmetric hyperbolic monopoles by attacking the problem directly via the geometry of spectral curves. Although our methods apply more generally, we will restrict ourselves to monopoles with the rotational symmetry of a platonic solid, and would like to construct associated spectral curves with arbitrary mass. The platonic solids we consider are the tetrahedron, the octahedron and the icosahedron, with (special) symmetry groups isomorphic to A_4 , S_4 and A_5 , respectively. Instead of the octahedron and the icosahedron, we could have taken the dual solids (cube and dodecahedron), which have the same group of symmetries.

We start by writing down Ansätze for real (k, k) curves in Z with the symmetry of a given platonic solid. Again, we may restrict ourselves to centred curves without losing generality. The following elementary observation will be useful:

Lemma 5.1. Let $\Sigma \subset Z$ be a centred (k, k) curve. If $(w, z) \in \Sigma$ is fixed by a rotation preserving $(0, 0, 1)$, then $w = z$.

Proof. The set of points of H^3 fixed by a (nontrivial) rotation in $\text{Stab}_{(0,0,1)}\text{PSL}_2\mathbb{C} \cong \text{SO}(3)$ is a geodesic through $(0, 0, 1)$. Lemma 2.1 then implies that the spectral lines fixed by this rotation must lie on the diagonal $\mathbb{P}_\Delta^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$. \square

Suppose that a platonic group $G \subset \text{Stab}_{(0,0,1)}\text{PSL}_2\mathbb{C} \cong \text{SO}(3)$ has been fixed; a pictorial way of doing this is to map the corresponding solid onto the star at $(0, 0, 1)$ (which we can think of as a two-sphere centred at $(0, 0, 1)$ and identify with \mathbb{P}_Δ^1), using central projection. In the following, we shall be interested in G -symmetric (k, k) curves Σ for which the space of orbits Σ/G is an elliptic curve. Note that, if Σ is smooth, so too will be E .

Proposition 5.2. Let $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth (k, k) curve invariant under $G = A_4, S_4$ or A_5 symmetry with quotient $\Sigma/G = E$ an elliptic curve. Then $k = 3$ or 4 in the $G = A_4$ case, and $k = 4$ and 6 in the S_4 , respectively A_5 cases.

Remark. In the course of the proof we will see that, together with the diagonal $\mathbb{P}_\Delta^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$, the curves satisfying the conditions of the proposition are the smallest degree smooth (k, k) curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with tetrahedral, octahedral or icosahedral symmetry.

Proof. The quotient Σ/G parametrises the orbits of G on $\Sigma \subset Z$. Generic orbits of G on \mathbb{P}_Δ^1 have $|G|$ points and thus the same is true of generic orbits of G on Z . There are

three exceptional orbits of G on \mathbb{P}_Δ^1 , and hence Z , given by orbits of vertices, (midpoints of) edges and (midpoints of) faces under the identification of \mathbb{P}_Δ^1 with the symmetric polyhedron. Thus generic G -orbits on a G -symmetric (k, k) curve Σ have $|G|$ points, and there are at most three exceptional orbits. In the tetrahedral case, there are three orbits consisting of 4, 6 and 4 points — the orbits of vertices, edges and faces — on the (k, k) curve Σ . For the octahedral and icosahedral cases, the exceptional orbits correspond to 6, 12 and 8 vertices, edges and faces, and 12, 30 and 20 vertices, edges and faces.

We say that any point in an exceptional orbit is an *exceptional point*. The Euler characteristic of Σ (given by $2k(2 - k)$) minus its exceptional points is divisible by $|G|$ since it admits a free G action:

$$|G| \mid 2k(2 - k) - \#\{\text{exceptional points}\}. \quad (55)$$

Any point of an exceptional orbit of G is fixed by some element of G and hence by Lemma 5.1 it lies in the diagonal $\mathbb{P}_\Delta^1 \subset Z$. Since we assume Σ to be smooth, it cannot contain the component \mathbb{P}_Δ^1 (except in the non-reduced case $\Sigma = k\mathbb{P}_\Delta^1$), so

$$\Sigma \cdot \mathbb{P}_\Delta^1 = (k, k) \cdot (1, 1) = 2k = \#\{\text{exceptional points}\} + \ell|G|$$

where the right-hand side consists of the exceptional orbits and of $\ell \geq 0$ orbits of size $|G|$. Thus

$$|G| \mid 2k(1 - k) \quad (56)$$

and we immediately deduce that $k = 0$ or $1 \bmod 3$, 4 or 5 in the respective tetrahedral, octahedral and icosahedral cases.

Since the quotient Σ/G has genus one, the Euler characteristic gives

$$\frac{2k(2 - k) - \#\{\text{exceptional points}\}}{|G|} = -\#\{\text{exceptional orbits}\} \geq -3. \quad (57)$$

In the tetrahedral case, $|G| = 12$, and if $k \geq 5$, then the left-hand side of (57) is less than -3 which contradicts the inequality, so for $1 < k < 5$, $k = 3$ and 4 give solutions of (56). The exceptional orbits consist of 6 points in the $k = 3$ case, and 4 plus 4 points in the $k = 4$ case to give a solution of (57). In the octahedral case, $|G| = 24$, so (57) implies that $k < 8$. Solutions of (56) must be 0 or $1 \bmod 4$, hence $k = 4$ or 5 and only 4 gives a solution of (56), with exceptional orbit of 8 points. In the icosahedral case, $|G| = 60$, so (57) implies that $k < 11$, and together with the mod 5 condition we need only check $k = 5, 6$ or 10 . Both $k = 6$ and $k = 10$ give solutions of (57), however only $k = 6$ with exceptional orbit of 12 points allows a solution of (57) and the proposition is proven. \square

It is useful to describe each of the exceptional G -orbits on \mathbb{P}_Δ^1 as the zeroes of a binary form — a homogeneous polynomial in the two homogeneous coordinates for \mathbb{P}_Δ^1 . One then obtains three forms $K_v, K_e, K_f \in \mathbb{C}[\zeta_0, \zeta_1]^{\text{hom}}$ describing the positions of the vertices, (midpoints of) edges and (midpoints of) faces of the corresponding polyhedron. G acts on \mathbb{P}_Δ^1 and this action can be transferred to the vector spaces of binary forms of each degree. By construction, K_e, K_f and K_v are projectively invariant under G , and for

each of these forms the scalar factors under elements of G give an abelian character of the platonic group. Since G is finite, suitable products of these three forms must be strictly invariant.

In his famous book [20] of 1884 on Galois theory, Felix Klein described the ring of G -invariant forms for the three platonic groups; in particular, the unique monic elements of minimal positive degree can be given in each case as follows, in an obvious orientation of the polyhedra:

- For $G = A_4$, $K_e(\zeta_0, \zeta_1) = \zeta_0 \zeta_1 (\zeta_1^4 - \zeta_0^4)$;
- For $G = S_4$, $K_f(\zeta_0, \zeta_1) = \zeta_1^8 + 14 \zeta_0^4 \zeta_1^4 + \zeta_0^8$;
- For $G = A_5$, $K_v(\zeta_0, \zeta_1) = \zeta_0 \zeta_1 (\zeta_1^{10} + 11 \zeta_0^5 \zeta_1^5 - \zeta_0^{10})$; in this case, all projectively invariant forms are strictly invariant, a consequence of simplicity.

Given the result of Lemma 5.1, candidates for smooth G -invariant (k, k) curves Σ of lowest k can now be written as

$$(w - z)^k + \alpha \tilde{K}(w, z) = 0, \quad (58)$$

where $\tilde{K}(w, z) \in \mathbb{C}[w, z]$ can be polarised to a G -invariant element of $(\mathbb{C}[w_0, w_1] \otimes \mathbb{C}[z_0, z_1])^{\text{hom}}$ that projects to the G -invariant forms $K_{e,f,v}$ above under the dual ι^* of the inclusion $\iota : \mathbb{P}_\Delta^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. The value of k corresponds to half of the degree of Klein's invariant form on \mathbb{P}_Δ^1 (i.e., $k = 3, 4$ or 6), since ι^* projects (p, q) -forms onto $(p + q)$ -forms. Thus for each G , this procedure yields the symmetric curves of lowest k in Proposition 5.2.

One way to find the appropriate function \tilde{K} on $\mathbb{P}^1 \times \mathbb{P}^1$ from Klein's invariant forms on \mathbb{P}_Δ^1 is to start with a polynomial Ansatz incorporating invariance under σ_+ in (18) and impose G -invariance, as illustrated in the following example. A reason why we should give σ_+ the chance of being an extra symmetry of \tilde{K} is that the parity transformation $\mathcal{P}_{(0,0,1)}$ in (20) leaves the zero-set of the particular Klein form associated with each platonic group invariant; notice that this is not true in general for the other two forms associated with a given platonic solid.

Example 5.3. Let $G = A_4$. In terms of the inhomogeneous coordinate $\zeta := \zeta_1/\zeta_0$, Klein's form $K_v(\zeta_0, \zeta_1)$ corresponds to the polynomial $\zeta(\zeta^4 - 1)$. The projection ι^* is described by its action on generators $\iota^*(w) = \iota^*(z) = \zeta$, thus we write

$$\tilde{K}_v(w, z) = \left(\frac{w+z}{2}\right) \left(c_1(wz)^2 + c_2(wz) \left(\frac{w+z}{2}\right)^2 + (1 - c_1 - c_2) \left(\frac{w+z}{2}\right)^4 - 1\right).$$

In the orientation chosen, A_4 is generated by the rotations

$$(w, z) \mapsto (-w, -z), \quad (w, z) \mapsto \left(\frac{1}{w}, \frac{1}{z}\right), \quad (w, z) \mapsto \left(\frac{w-i}{w+i}, \frac{z-i}{z+i}\right). \quad (59)$$

Imposing invariance under these, one finds $c_1 = 1$ and $c_2 = 0$. Thus we write the Ansatz (58) as

$$(w - z)^3 + i\alpha(w + z)((wz)^2 - 1) = 0. \quad (60)$$

An easy check shows that this curve is real with respect to (7) if and only if $\alpha \in \mathbb{R}$.

An easy check shows that the curve (60) is real with respect to (7) if and only if $a \in \mathbb{R}$. The argument we used in section 4.3 shows that $m = 0$ corresponds to $\alpha = \sqrt{3}$, given that the rational map of degree 3 with the A_4 -symmetry that we are using is [23]

$$R(z) = \frac{1 - i\sqrt{3}z^2}{i\sqrt{3}z - z^3}.$$

Moreover, $m \rightarrow \infty$ should correspond to $\alpha = 0$ (as the limit spectral curve will be $\text{SO}(3)$ -symmetric, thus three copies of the diagonal \mathbb{P}_Δ^1). Thus in fact we can take

$$0 < \alpha < \sqrt{3}$$

and it is easy to show that (60) is smooth and irreducible for all these values of α .

Using the same technique, we can find the corresponding polynomial \tilde{K}_f for $G = S_4$ and write (58) as

$$(w - z)^4 + \alpha(w^4 z^4 + 6w^2 z^2 + 4wz(w^2 + z^2) + 1) = 0. \quad (61)$$

This curve lies in the 2-dimensional family of A_4 -symmetric degree (4, 4) curves

$$(w - z)^4 + \alpha(w^4 z^4 + 6w^2 z^2 + 4wz(w^2 + z^2) + 1) + i\beta(w - z)(w + z)((wz)^2 - 1) = 0 \quad (62)$$

which are real provided $\alpha, \beta \in \mathbb{R}$. It is easy to check that (62) are all invariant under the rotations (59), and if $\beta = 0$ also under the extra order four rotation

$$(w, z) \mapsto (iw, iz). \quad (63)$$

In fact, the curves (62) are still invariant under

$$(w, z) \mapsto (iz, iw), \quad (64)$$

which together with (59) generates S_4 , and then Σ/S_4 is also an elliptic curve, double-covered by Σ/A_4 ; however, since (64) is not a rotation we do not call this S_4 an octahedral symmetry group. Note that the curves (60) in the $k = 3$ case are also invariant under this hidden S_4 symmetry, but in this case Σ/S_4 is a rational curve. As for $k = 3$, the limit $m \rightarrow \infty$ of (60) should give $\alpha = 0$, and the value of α for $m = 0$ can be calculated from the S_4 -invariant rational map [23]

$$R(z) = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}$$

to be $\alpha = 1$, thus we take

$$0 < \alpha < 1;$$

again, Σ is smooth and irreducible for all these values of α .

For $G = A_5$ we have one parameter $\alpha \in \mathbb{R}$ in

$$(w - z)^6 + \alpha(9(w^6 + z^6) + 9wz(w^4 + z^4) + 10(wz)^2(w^2 + z^2) + 10(wz)^3 + 3(w + z)((wz)^5 - 1)) = 0. \quad (65)$$

This final example fails the vanishing cohomology condition 3. (cf. Propositions 8.1 and 8.2). As in the euclidean case, it is necessary to multiply the degree 6 polynomial by $(w - z)$ to obtain a reducible degree $(7, 7)$ curve, so that the holomorphic sections on the degree $(6, 6)$ component have to satisfy enough further conditions to be forced to vanish. We will not treat this reducible spectral curve here.

6 Tetrahedral and octahedral symmetry

In the rest of the paper, we shall relate the symmetric (k, k) curves above to spectral curves of hyperbolic monopoles with a given mass. Our main aim is to obtain the mass associated with these curves, and the basic strategy will consist of transferring calculations on Σ to the quotients by their platonic symmetry groups,

$$\pi : \Sigma \rightarrow \Sigma/G =: E. \quad (66)$$

In this way, the complex analysis needed to relate the monopole mass m to the parameter α in the Ansätze will have the same flavour as the $k = 2$ discussion above. However, unlike the 2-monopole case, we will now have to deal with a nontrivial condition 3., which we shall approach using some classical algebraic geometry. It will turn out that condition 3. will now also play a rôle in the mass calculation itself. Our results can be summarised as follows:

Theorem 6.1. There is a unique $\mathrm{PSL}_2\mathbb{C}$ -orbit of tetrahedrally symmetric 3-monopoles with mass $m > 0$; a representative is the centred monopole with spectral curve (60), where α and m satisfy the relation

$$\wp\left(\frac{2\varpi_1}{2m+3} \middle| \varpi_1, \varpi_2\right) = \frac{1}{12} - \frac{1}{\alpha^2} \quad (67)$$

involving the Weierstraß \wp -function of the elliptic curve with invariants (83).

Theorem 6.2. There is a unique $\mathrm{PSL}_2\mathbb{C}$ -orbit of octahedrally symmetric 4-monopoles with representative (61), whose mass is determined by the relation

$$\wp\left(\frac{3\varpi_1}{m+2} \middle| \varpi_1, \varpi_2\right) = \frac{-4\alpha^4 + 10\alpha^3 - 115\alpha^2 + 60\alpha - 3}{54\alpha^2(\alpha+1)^2}, \quad (68)$$

where the Weierstraß \wp -function has invariants (93) and (94).

In both cases, we actually construct a curve together with a linear flow on its jacobian variety that avoids the theta-divisor; this will be discussed in sections 7 and 8. By quite general results (see [1, 10, 12]) this gives rise to a Lax system which we refer to as a monopole.

We express the relation between α and m for each Ansatz in terms of the 1-parameter family of elliptic curves $E \equiv E_\alpha$; thus the half-periods ϖ_1 and ϖ_2 in equations (67) and (68), which we shall specify carefully below, are functions of α . To obtain this relation,

our starting point is a reciprocity argument similar to the one in section 3.2. We let ξ_j denote trivialising sections of $L^{2m+k}|_\Sigma$ over the open sets $U_j \cap \Sigma$ ($j = 1, 2$) defined by (9), and use them to obtain differentials of the third kind $\omega_j = d \log \xi_j$ on Σ , related through

$$\omega_1 = (2m + k) \left(\frac{dw}{w} - \frac{dz}{z} \right) + \omega_2.$$

We want to apply the reciprocity law generalising (40) to ω_1 and to $\pi^*\omega$, where ω is a global holomorphic 1-form on E ; obviously, $\pi^*\omega$ is a differential of the first kind on Σ which is invariant under the G -action. Fix a canonical basis $\{a_\ell, b_\ell\}_{\ell=1}^g$ for $H_1(\Sigma; \mathbb{Z}) \cong \mathbb{Z}^{2g}$, where $g = (k-1)^2$, and define integers m_ℓ, n_ℓ by

$$m_\ell := \frac{1}{2\pi i} \oint_{b_\ell} \omega_1, \quad n_\ell := -\frac{1}{2\pi i} \oint_{a_\ell} \omega_1.$$

We denote by p_j, q_j the poles of $\frac{dw}{w} - \frac{dz}{z}$; they can be found explicitly using the equation for Σ . Our conventions are such that the residues of ω_1 at these poles are given by

$$\text{Res}_{p_j}(\omega_1) = -(2m + k) \quad \text{and} \quad \text{Res}_{q_j}(\omega_1) = 2m + k \quad j = 1, \dots, 2(k-1).$$

Using the explicit description in section 10, we shall see that in both cases the points p_j and q_j lie on two separate G -orbits, and we set

$$p := \pi(p_j), \quad q := \pi(q_j).$$

The reciprocity law can now be written as

$$\sum_{\ell=1}^g \begin{vmatrix} \oint_{a_\ell} \pi^*\omega & -n_\ell \\ \oint_{b_\ell} \pi^*\omega & m_\ell \end{vmatrix} = (2m + k) \sum_{j=1}^{2(k-1)} \int_{p_j}^{q_j} \pi^*\omega, \quad (69)$$

where the paths of integration in the right-hand side avoid the 1-homology basis.

Let $I =]0, \sqrt{3}[$ if $k = 3$, respectively $I =]0, 1[$ if $k = 4$, denote the range of α . Equation (69) determines $\alpha \in I$ as a function of m , once the integers m_ℓ, n_ℓ are known. In fact, these integers are constant along the isotopy of curves $\Sigma = \Sigma_\alpha$ given by (60) or (61):

Lemma 6.3. The integers m_ℓ, n_ℓ defined by equation (69) are independent of $\alpha \in I$.

Proof. Suppose that α and m are fixed. For each element ω_i of a basis of global holomorphic 1-forms on Σ , a reciprocity relation like (69) can be written, where $\pi^*\omega$ is replaced by ω_i . Taking real and imaginary parts of these g equations, one obtains $2g$ real equations in the $2g$ real unknowns m_ℓ, n_ℓ . If the basis $\{\omega_i\}_i$ is dual to the 1-cycles a_ℓ , then we see that the equations corresponding to the imaginary parts decouple the variables n_ℓ , and we can solve for them as the matrix of coefficients is nonsingular by the second Riemann bilinear relations [11]. Substitution in the equations corresponding to the real parts give immediately the m_ℓ , as the matrix of coefficients is then the identity by construction. We conclude that we can always solve for m_ℓ, n_ℓ , and the solutions are given as linear

fractional functions of the periods of Σ and the real and imaginary parts of the integrals $\int_{q_j}^{p_j} \omega_i$.

We now argue that the periods of Σ are continuous functions of α . Let $\{\varphi_\alpha\}_{\alpha \in I}$ be an isotopy of Z such that $\varphi_\alpha(\Sigma_{1/2}) = \Sigma_\alpha$ for all $\alpha \in I$, say. Fix representatives of a basis of 1-cycles $\{c_\ell\}_\ell$ on $\Sigma_{1/2}$, and transfer them to each Σ_α using φ_α . Moreover, equip each curve with a basis of global holomorphic 1-forms ω_i^α obtained by adjunction (i.e. taking Poincaré residues of a fixed set of g holomorphic 2-forms on a neighbourhood of the family of curves in Z); this is possible as each Σ_α is smooth. Then one finds

$$\oint_{\varphi_{\alpha*} c_\ell} \omega_i^\alpha = \oint_{c_\ell} \varphi_\alpha^* \omega_i^\alpha$$

and the right-hand side is an integral of a continuous function of α , which is itself continuous in α . Clearly, the same type of argument works for the paths connecting poles of $\frac{dw}{w}$ to poles of $\frac{dz}{z}$, and the lemma follows. \square

To transfer the calculation to E , we define

$$c := \pi_* \left(\sum_{j=1}^g (m_j a_j + n_j b_j) \right) =: \ell_1 a + \ell_2 b, \quad (70)$$

where $\{a, b\}$ is a standard basis of $H_1(E; \mathbb{Z})$, and obtain

$$\oint_c \omega = 2(2m + k) \int_p^q \omega \quad (71)$$

after using π to change variables and clearing a factor of $\deg \pi = |G|$. As a consequence of Lemma 6.3, the components of c will not change in an isotopy of bases of $H_1(E; \mathbb{Z})$ defined for $\alpha \in A$. Note that, although the paths connecting q_j to p_j on Σ had to be chosen to have zero intersection with the elements of $H_1(\Sigma; \mathbb{Z}) \hookrightarrow H_1(\Sigma, \{p_j, q_j\}; \mathbb{Z})$, this does not necessarily hold as we project to E . However, there is a simple criterion to test whether a path γ from q to p on E is the image of a suitable path on Σ , namely, the components of the 1-cycle $c \in H_1(E; \mathbb{Z})$ must be independent of m for solutions (α, m) of (71); once at least two such solutions are obtained, γ can be found systematically by expanding in a basis of $H_1(E, \{p, q\}; \mathbb{Z}) \cong \mathbb{Z}^3$. In section 9, we shall describe a general procedure that allows one to obtain data (α, m) for half-integer values of m . From these, we will be able to calculate both a suitable path γ and the 1-cycle c from an explicit construction of E in section 10. Finally, we will use uniformisation on E to evaluate the mass constraint (71) in terms of elliptic functions.

7 Canonical embedding

The canonical embedding of a genus g curve Σ is a map

$$\Sigma \rightarrow \mathbb{P}^{g-1}$$

defined by $z \mapsto [\omega_1(z) : \dots : \omega_g(z)]$ with respect to a local trivialisation of the canonical line bundle K_Σ . This map is an embedding except when the curve is hyperelliptic, in which case it maps 2-to-1 onto a rational curve in \mathbb{P}^{g-1} . It gives a useful geometric version of Riemann–Roch which we will use.

By adjunction, the canonical embedding of a curve embedded in a surface X can be induced by an embedding $X \rightarrow \mathbb{P}^{g-1}$. We will use this to give an explicit description of the canonical embedding of a curve in the quadric $Q := \mathbb{P}^1 \times \mathbb{P}^1$. (This can be done similarly for \mathbb{P}^2 . An immediate consequence is that no smooth hyperelliptic curve with $g > 1$ embeds in \mathbb{P}^2 or $\mathbb{P}^1 \times \mathbb{P}^1$.)

Lemma 7.1. For $k > 2$, the canonical embedding of a smooth (k, k) curve $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1$ is the composition

$$\begin{aligned} \Sigma &\hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{k(k-2)} \\ (w, z) &\mapsto [1 : w : \dots : w^{k-2} : z : wz : \dots : w^{k-2}z : \dots : z^{k-2} : wz^{k-2} : \dots : w^{k-2}z^{k-2}] \end{aligned}$$

Proof. As mentioned above we will use the adjunction formula $K_\Sigma = \mathcal{O}(k-2, k-2)|_\Sigma$. Sections of K_Σ can be identified with sections that extend to all of $Q = \mathbb{P}^1 \times \mathbb{P}^1$ since the relation introduced by the equation of Σ is in higher degree than $k-2$, or equivalently the outer two cohomology groups vanish in the following exact sequence:

$$H^0(Q, \mathcal{O}(-2, -2)) \rightarrow H^0(Q, \mathcal{O}(k-2, k-2)) \rightarrow H^0(\Sigma, \mathcal{O}(k-2, k-2)) \rightarrow H^1(Q, \mathcal{O}(-2, -2)).$$

The polynomials $1, w, \dots, w^{k-2}, z, wz, \dots, w^{k-2}z, \dots, z^{k-2}, wz^{k-2}, \dots, w^{k-2}z^{k-2}$ give a basis of the space of sections of $\mathcal{O}(k-2, k-2)$, where we are using U_1 in (9) as a local trivialising set. \square

Lemma 7.2. For any smooth $(3, 3)$ curve $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1$, a nontrivial linear equivalence relation

$$p_1 + p_2 + p_3 \sim q_1 + q_2 + q_3$$

is equivalent to $p_1 + p_2 + p_3 \sim \mathcal{O}_\Sigma(1, 0)$ or $\mathcal{O}_\Sigma(0, 1)$.

Proof. This is a well-known fact. Clearly any two fibres of the projection of $\mathbb{P}^1 \times \mathbb{P}^1$ to the first factor are linearly equivalent, and the same for projection to the second factor. It ends up that these are the only linear equivalences between triples of points on a smooth degree $(3, 3)$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$. An equivalent statement is that a g_3^1 on Σ must be $\mathcal{O}_\Sigma(1, 0)$ or $\mathcal{O}_\Sigma(0, 1)$. (Recall that a g_d^r is any linear system of degree d and dimension $r+1$, see for example [2].) We include the proof as a warm-up for similar results.

We use the canonical embedding

$$\phi : \Sigma \rightarrow \mathbb{P}^3$$

to prove this. The images $\phi(p_1)$, $\phi(p_2)$ and $\phi(p_3)$ lie inside a $\mathbb{P}^2 \subset \mathbb{P}^3$. When there exists the nontrivial relation $p_1 + p_2 + p_3 \sim q_1 + q_2 + q_3$, the geometric version of Riemann–Roch says that $\phi(p_1)$, $\phi(p_2)$ and $\phi(p_3)$ lie inside a $\mathbb{P}^1 \subset \mathbb{P}^3$.

Suppose two of the points lie on a $(1, 0)$ or $(0, 1)$ curve, say p_1 and p_2 lie on a $(1, 0)$ curve. Then their images determine a line in \mathbb{P}^3 , $[1 : w : c : wc]$ for some constant c . The image of p_3 , given by $[1 : w : z : wz]$, lies on the curve only if $z = c$, i.e. p_3 lies on the same $(1, 0)$ curve, proving the lemma.

If no two points lie on a $(1, 0)$ or $(0, 1)$ curve, then the three points lie on a smooth $(1, 1)$ curve. Any smooth $(1, 1)$ curve is equivalent to the diagonal $\mathbb{P}_\Delta^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$ which has images $[1 : w : w : w^2]$ inside \mathbb{P}^3 , and any three different points of this form are linearly independent due to the $1, w, w^2$ terms. This contradicts the fact that the points p_1, p_2 and p_3 span \mathbb{P}^1 . Essentially the same argument is used when some of the p_i coincide, so the result follows. \square

Recall that a divisor D is *special* if $H^1(D) \neq 0$, or equivalently if it has more sections than a generic divisor of the same degree.

Lemma 7.3. A positive divisor D of degree at most 7 on a $(4, 4)$ curve $\Sigma \subset Q$ is special precisely when one of the following holds:

- (a) D contains four points on a $(1, 0)$ or $(0, 1)$ curve;
- (b) D contains six points on a $(1, 1)$ curve.

Proof. The proof of this requires a systematic analysis of many separate cases. To avoid this we will instead refer to an exercise from [2], p. 199, stating that any collection of at most seven points in \mathbb{P}^3 that fails to impose independent conditions on quadrics contains one of the following: (i) four collinear points; (ii) six points on a conic; (iii) seven coplanar points.

Lemma 7.1 shows that the canonical embedding of Σ factors through the quadric Q . More is true. It also factors through \mathbb{P}^3 , so we have

$$\Sigma \hookrightarrow Q \hookrightarrow \mathbb{P}^3 \hookrightarrow \mathbb{P}^8$$

with rightmost map $[z_0 : z_1 : z_2 : z_3] \mapsto [z_0^2 : z_0 z_1 : z_0 z_2 : z_0 z_3 : z_1^2 : z_1 z_3 : z_2^2 : z_2 z_3 : z_3^2]$ the degree two Veronese map, where the monomial $z_1 z_2$ is missing since it is equal to $z_0 z_3$ on Q . Thus a collection of points on Σ representing a positive divisor D gives a collection of points in \mathbb{P}^3 . By the geometric version of Riemann–Roch, D is special when the images of the points in \mathbb{P}^3 are dependent, so one of (i), (ii), or (iii) occurs.

The intersection of a line $L \subset \mathbb{P}^3$ and Q either consists of two points (counted with multiplicity) or $L \subset Q$. If (i) occurs then four points from Q lie on a line $L \subset \mathbb{P}^3$ and hence $L \subset Q$. The only such lines are $(1, 0)$ or $(0, 1)$ curves so case (a) holds.

The intersection of a conic $C \subset \mathbb{P}^3$ and Q either consists of four points or $C \subset Q$. If (ii) occurs, then six points from Q lie on a conic $C \subset \mathbb{P}^3$ and hence $C \subset Q$. Conics in Q are $(1, 1)$ curves so (b) holds. The intersection of a plane and the quadric in \mathbb{P}^3 is a conic in \mathbb{P}^2 and a $(1, 1)$ curve in Q , so if (iii) occurs then seven points lie on a $(1, 1)$ curve and again (b) holds. \square

Lemma 7.4. For any smooth $(4, 4)$ curve $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1$, a nontrivial linear equivalence relation

$$p_1 + p_2 + p_3 + p_4 \sim q_1 + q_2 + q_3 + q_4$$

implies $p_1 + p_2 + p_3 + p_4 \sim \mathcal{O}_\Sigma(1, 0)$ or $\mathcal{O}_\Sigma(0, 1)$.

Proof. This is immediate from Lemma 7.3. \square

Lemma 7.5. For any smooth $(4, 4)$ curve $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1$, the existence of two independent linear equivalence relations

$$p_1 + \dots + p_8 \sim q_1 + \dots + q_8 \sim r_1 + \dots + r_8$$

implies that there exists p_9, \dots, p_{12} such that $p_1 + \dots + p_8 + p_9 + \dots + p_{12} \sim \mathcal{O}_\Sigma(2, 1)$ or $\mathcal{O}_\Sigma(1, 2)$.

Proof. By *independent* relations we mean that the points satisfy one of the three equivalent conditions: $h^0(p_1 + \dots + p_8) > 2$; or $p_1 + \dots + p_8$ defines a g_8^2 ; or the images of p_1, \dots, p_8 in \mathbb{P}^8 live in a \mathbb{P}^5 . Since the images of p_1, \dots, p_8 in \mathbb{P}^8 satisfy two relations, any subset of seven points satisfies a relation and we can apply Lemma 7.3 to all eight such subsets.

(I) If seven points contain six points on a smooth $(1, 1)$ curve, then no four points of the eight points are contained on a line, so all subsets of seven points must contain six points in a conic. Since a conic is determined by three points, the eight conics must coincide. Thus the eight points consist of seven points on a smooth conic and an eighth general point. In particular, the eight points lie inside a $(2, 1)$ or $(1, 2)$ curve as claimed.

(II) If four of the eight points are contained on a line, say a $(1, 0)$ curve, then since five points cannot lie on a line — Σ is a smooth $(4, 4)$ curve —, when a subset of seven points does not include these four points, it must include either four points on another line, or three points on a $(0, 1)$ curve so that it has $3 + 3$ points on a reducible $(1, 1)$ curve. In both these cases, seven points lie on two lines, and the eighth point is general, so in particular the eight points lie inside a $(2, 1)$ or $(1, 2)$ curve as claimed.

(III) If neither (I) nor (II), i.e. no four points lie on a line and no six points lie on a smooth $(1, 1)$ curve then case (b) of Lemma 7.3 must apply to all eight subsets, with a reducible $(1, 1)$ curve, i.e. $3 + 3$ points must lie on a $(1, 0)$ and $(0, 1)$ curve. A subset of seven points may not contain the three points on the $(1, 0)$ curve, so another $(1, 0)$ curve must contain three points, and this takes at least two extra points from the eight points, i.e. $3 + 2 + 3$ points distributed on a $(1, 0)$, $(1, 0)$ and $(0, 1)$ curve. Similarly we also need another $(0, 1)$ curve with three points and this requires at least one more point, so nine points are required. Thus, case (III) is empty and the lemma is proven. \square

8 Crossing the theta-divisor

8.1 $k = 3$ tetrahedral symmetry

Consider the smooth genus four curve $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1$ with equation (60) for $\alpha \in \mathbb{R}^+$. The bundle $L^s(1, 0)|_\Sigma$ is a degree 3 bundle, or equivalently a degree 3 divisor on Σ . Recall that the divisor D of $L^s(1, 0)|_\Sigma$ meets the theta-divisor if one of the following equivalent properties holds:

1. $D \sim p_1 + p_2 + p_3$ i.e. D is linearly equivalent to a positive divisor;

2. there exists a meromorphic function f on Σ such that $(f) + D \geq 0$;
3. there exists a nontrivial holomorphic section of the line bundle $L^s(1, 0)|_\Sigma$;
4. $H^0(\Sigma, L^s(1, 0)) \neq 0$.

Here we prove that, when $0 < s < 2m + 2$, the divisor of $L^s(1, 0)|_\Sigma$ *does not* meet the theta-divisor.

Proposition 8.1. $H^0(\Sigma, L^s(1, 0)) = 0$ for $0 < s < 2m + 2$.

Proof. The geometry implies that L is invariant under the A_4 -action, since L comes from the standard $U(1)$ monopole which is symmetric under $SO(3)$. In other words, $\mathcal{O}(1, -1)$ is invariant under the $SO(3)$ action on $\mathbb{P}^1 \times \mathbb{P}^1$ so $L_\Sigma := \mathcal{O}(1, -1)|_\Sigma$ is invariant under A_4 . In particular, L_Σ is the pull-back of a degree zero line bundle \hat{L} on $E = \Sigma/A_4$, and $L_\Sigma^s = \pi^* \hat{L}^s$ for $s \in \mathbb{C}$. Since E is an elliptic curve, as a divisor $\hat{L}^s \sim p - p'$, and all such divisors are represented as s varies over the complex numbers. Thus, since \hat{L}^s pulls back to L_Σ^s , we can represent the divisor of any L_Σ^s by

$$L_\Sigma^s \sim \text{Orb}_p - \text{Orb}_{p'}$$

where Orb_p is the A_4 -orbit in Σ that lies over $p \in E$. So questions involving L_Σ^s are questions about orbits of A_4 .

We will show that the difference of two orbits of A_4 in Σ plus the divisor class $\mathcal{O}_\Sigma(1, 0)$ lies in the theta-divisor,

$$\text{Orb}_p - \text{Orb}_{p'} + \mathcal{O}_\Sigma(1, 0) \sim q_1 + q_2 + q_3, \quad (72)$$

in the following trivial cases:

$$\text{Orb}_p - \text{Orb}_{p'} \sim 0 \quad \text{or} \quad \text{Orb}_p - \text{Orb}_{p'} \sim L_\Sigma^{-1} = \mathcal{O}_\Sigma(-1, 1).$$

The action of A_4 on the left hand side of (72) preserves the two orbits and also preserves the linear equivalence class $\mathcal{O}_\Sigma(1, 0)$ since the action, given in (59), preserves the two \mathbb{P}^1 factors of $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, for any $g \in A_4$,

$$q_1 + q_2 + q_3 \sim gq_1 + gq_2 + gq_3. \quad (73)$$

The collection $\{q_1, q_2, q_3\}$ cannot be invariant under A_4 , since the orbits of A_4 have size 12 and 6. Thus we can choose a $g \in A_4$ that does not preserve $\{q_1, q_2, q_3\}$, so the linear equivalence relation (73) is a nontrivial relation between degree 3 positive divisors. By Lemma 7.2, this implies one of the two cases

$$q_1 + q_2 + q_3 \sim \mathcal{O}_\Sigma(1, 0) \quad \text{or} \quad q_1 + q_2 + q_3 \sim \mathcal{O}_\Sigma(0, 1).$$

If $q_1 + q_2 + q_3 \in \mathcal{O}_\Sigma(1, 0)$, then (72) reduces to

$$\text{Orb}_p - \text{Orb}_{p'} \sim 0,$$

in other words L_Σ^s is trivial, so $s = 0$ (or a multiple of $2m + 3$ since $L_\Sigma^{2m+3} \cong \mathcal{O}_\Sigma$.) If $q_1 + q_2 + q_3 \in \mathcal{O}_\Sigma(0, 1)$, then

$$0 \sim \text{Orb}_p - \text{Orb}_{p'} + \mathcal{O}_\Sigma(1, 0) - (q_1 + q_2 + q_3) \sim L_\Sigma^{s+1},$$

so $s = -1$ (plus a multiple of $2m + 3$.) In particular, when $0 < s < 2m + 2$, $L^s(1, 0)|_\Sigma$ does not meet the theta-divisor. \square

8.2 $k = 4$ octahedral symmetry

Orbits of S_4 on Σ consist of 24 points, except for the one exceptional orbit of 8 points given by the $(1, 1)$ divisor $\mathbb{P}_\Delta^1 \cap \Sigma$.

Proposition 8.2. $H^0(\Sigma, L^s(2, 0)) = 0$ for $0 < s < 2m + 2$.

Proof. As before, we reduce questions involving L_Σ^s to questions about orbits of S_4 using

$$L_\Sigma^s \sim \text{Orb}_p - \text{Orb}_{p'}$$

for $p, p' \in E = \Sigma/S_4$.

The difference of two orbits of S_4 in Σ plus a $\mathcal{O}_\Sigma(2, 0)$ divisor lies in the theta-divisor,

$$\text{Orb}_p - \text{Orb}_{p'} + \mathcal{O}_\Sigma(2, 0) \sim q_1 + q_2 + \dots + q_8, \quad (74)$$

in the following trivial cases:

$$\text{Orb}_p - \text{Orb}_{p'} \sim L_\Sigma^\epsilon, \quad \epsilon = 0, -1 \text{ or } -2. \quad (75)$$

The action of S_4 on the left-hand side of (74) preserves the two orbits and also preserves the linear equivalence class $\mathcal{O}(2, 0)$. Thus, for any $g \in S_4$,

$$q_1 + q_2 + \dots + q_8 \sim gq_1 + gq_2 + \dots + gq_8. \quad (76)$$

If the collection $\{q_1, q_2, \dots, q_8\}$ is invariant under S_4 then it consists of the exceptional orbit and

$$q_1 + q_2 + \dots + q_8 \sim \mathcal{O}_\Sigma(1, 1),$$

which yields $\epsilon = -1$ in (75).

If the collection $\{q_1, q_2, \dots, q_8\}$ is not invariant under S_4 , then we can choose a $g \in S_4$ that does not preserve $\{q_1, q_2, \dots, q_8\}$, so the linear equivalence relation (76) is a nontrivial relation between degree 8 positive divisors, or equivalently $\dim H^0(\Sigma, L^s(2, 0)) \geq 2$. We may assume that there is no section in $H^0(\Sigma, L^s(2, 0))$ that is invariant under S_4 , or more generally generates a 1-dimensional representation of S_4 , since the zero set of such a section would have to be the exceptional orbit of S_4 and we get the previous case of $\epsilon = -1$. Thus, if $H^0(\Sigma, L^s(2, 0)) = \mathbb{C}^2$ it must be an irreducible representation of S_4 . The standard 2-dimensional representation is given by the action on $H^0(\Sigma, \mathcal{O}(1, 0))$ and thus the tensor product $H^0(\Sigma, L^s(3, 0))$ is the 4-dimensional representation with a 1-dimensional irreducible component generated by a section $\xi = w\chi \in H^0(\Sigma, L^s(3, 0))$ for

$\chi \in H^0(\Sigma, L^s(2, 0))$. The zero set of ξ is invariant under S_4 and contains a $(1, 0)$ divisor. But this is impossible simply because any orbit of S_4 , or collection of exceptional orbits, contains at most three points in a $(1, 0)$ curve.

Thus $\dim H^0(\Sigma, L^s(2, 0)) > 2$, so by Lemma 7.5 there exist points $q_9, q_{10}, q_{11}, q_{12}$ such that

$$q_1 + q_2 + \dots + q_{12} \sim \mathcal{O}_\Sigma(2, 1) \text{ or } \mathcal{O}_\Sigma(1, 2) \quad (77)$$

and (74) becomes

$$\text{Orb}_p - \text{Orb}_{p'} + \mathcal{O}_\Sigma(2, 0) + q_9 + q_{10} + q_{11} + q_{12} \sim \mathcal{O}_\Sigma(2, 1) \text{ or } \mathcal{O}_\Sigma(1, 2).$$

Since $\text{Orb}_p - \text{Orb}_{p'} = L_\Sigma^s$ and any multiple of $\mathcal{O}_\Sigma(1, -1)$ is a power of L_Σ , we can adjust this expression and choose two new orbits \tilde{p} and \tilde{p}' so that

$$\text{Orb}_{\tilde{p}} - \text{Orb}_{\tilde{p}'} + \mathcal{O}_\Sigma(1, 0) \sim q_9 + q_{10} + q_{11} + q_{12}, \quad (78)$$

which is a reduction of the original problem. Again we use $g \in S_4$ to get a nontrivial linear equivalence $q_9 + q_{10} + q_{11} + q_{12} \sim gq_9 + gq_{10} + gq_{11} + gq_{12}$ and apply Lemma 7.4 to obtain

$$q_9 + q_{10} + q_{11} + q_{12} = \mathcal{O}_\Sigma(1, 0) \text{ or } \mathcal{O}_\Sigma(0, 1).$$

Put this back into (77) to get

$$q_1 + q_2 + \dots + q_8 = \mathcal{O}_\Sigma(2, 0), \mathcal{O}_\Sigma(1, 1), \text{ or } \mathcal{O}_\Sigma(0, 2)$$

and hence

$$\text{Orb}_p - \text{Orb}_{p'} \sim L_\Sigma^\epsilon, \epsilon = 0, -1 \text{ or } -2.$$

In particular, we have proved that when $0 < s < 2m + 2$, $L^s(2, 0)|_\Sigma$ does not meet the theta-divisor. \square

Remark. Hitchin [13] proves a result analogous to Proposition 8.2 in the euclidean case using a slightly different method. His treatment of the tetrahedral case is essentially the same as ours, relying on knowledge of g_3^1 divisors on a genus four curve. Whereas we extend this approach through an analysis of g_4^1 and g_8^2 divisors on a genus nine curve, Hitchin analyses the representation theory of S_4 more thoroughly and produces a beautiful application of the McKay correspondence. We need a small amount of representation theory to exclude a two-dimensional representation of S_4 , mainly because the space of g_8^1 divisors is too large to manage. The family of tetrahedrally symmetric $(4, 4)$ curves (62) cannot be treated using the representation theory approach, while the method here does generalise.

9 Half-integer mass

We have shown that triviality of L^{2m+k} on Σ determines a relation between α and m . The vanishing of $H^0(\Sigma, L^{2m}(k-2, 0))$ also has a direct consequence on the parameter α of the symmetric spectral curve. Since

$$H^0(\Sigma, L^{2m}(k-2, 0)) = H^0(\Sigma, \mathcal{O}(-2, k)),$$

a section can be described by

$$\xi_1(w, z) = w^{-2} z^k \xi_2(w^{-1}, z^{-1}) + \psi(w, z) \chi(w^{-1}, w, z^{-1}, z),$$

where ξ_1, ξ_2 give the section over the two coordinate patches U_1 and U_2 as in (9), $w^{-2} z^k = g_{12}(w, z)$ is the transition function of $\mathcal{O}(-2, k)$, i.e. its Čech cohomology class, and we work in the Laurent polynomial ring $\mathbb{C}[z, z^{-1}, w, w^{-1}]$ modulo $\psi(w, z)$, the defining polynomial of Σ .

A section exists if we can find $\chi(w^{-1}, w, z^{-1}, z)$ so that $\psi(w, z) \chi$ cancels all of the negative powers of w and z in $w^{-2} z^k \xi_2(w^{-1}, z^{-1})$. So $\psi(w, z)$ acts as a type of Toeplitz operator on χ . In fact $\chi = \chi(w^{-1}, w)$ and only the coefficients of w^{-1} need to be taken care of. This is not hard to prove, and is most efficiently expressed in terms of an exact sequence in Čech cohomology in which we see that q is also a Čech cocycle,

$$0 \rightarrow H^0(\Sigma, \mathcal{O}(-2, k)) \rightarrow H^1(Q, \mathcal{O}(-2 - k, 0)) \xrightarrow{\psi} H^1(Q, \mathcal{O}(-2, k));$$

so $\chi \in H^1(Q, \mathcal{O}(-2 - k, 0))$ and multiplication by $\psi(w, z)$ yields a linear map

$$\mathbb{C}^{k+1} \cong H^1(Q, \mathcal{O}(-2 - k, 0)) \xrightarrow{\psi} H^1(Q, \mathcal{O}(-2, k)) \cong \mathbb{C}^{k+1} \quad (79)$$

with kernel corresponding to sections of $H^0(\Sigma, \mathcal{O}(-2, k))$. Since $H^0(\Sigma, \mathcal{O}(-2, k))$ vanishes, the kernel is trivial and the determinant of (79) is nonzero. In the tetrahedral case (79) is represented by the matrix

$$\Psi = \begin{pmatrix} 1 & 0 & -i\alpha & 0 \\ 0 & 3 & 0 & i\alpha \\ i\alpha & 0 & 3 & 0 \\ 0 & -i\alpha & 0 & 1 \end{pmatrix}, \quad \det \Psi = (3 - \alpha^2)^2 \neq 0$$

and in the octahedral case by

$$\Psi = \begin{pmatrix} 1 & 0 & 0 & 0 & \alpha \\ 0 & 4 - 4\alpha & 0 & 0 & 0 \\ 0 & 0 & 6 + 6\alpha & 0 & 0 \\ 0 & 0 & 0 & 4 - 4\alpha & 0 \\ \alpha & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \det \Psi = 96(1 + \alpha)^2(1 - \alpha)^3 \neq 0.$$

The nonvanishing determinant restricts $0 < \alpha < \sqrt{3}$ in the tetrahedral case and $0 < \alpha < 1$ in the octahedral case, in agreement with the discussion in section 5.

More significant information is obtained from

$$0 \rightarrow H^0(\Sigma, \mathcal{O}(-3, k + 1)) \rightarrow H^1(Q, \mathcal{O}(-3 - k, 1)) \xrightarrow{\psi_1} H^1(Q, \mathcal{O}(-3, k + 1)),$$

where ψ_1 is again a Toeplitz type operator induced from multiplication by ψ . Trivial kernel, and hence nonzero determinant of Ψ_1 , occurs when $m > \frac{1}{2}$, since

$$H^0(\Sigma, L^{2m-1}(k - 2, 0)) = H^0(\Sigma, \mathcal{O}(-3, k + 1))$$

	$G = A_4$	$G = S_4$
m	$k = 3$	$k = 4$
0	$\sqrt{3}$	1
$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{3}$
1	$2 - \sqrt{3}$	$\frac{1}{7}$
$\frac{3}{2}$	$\sqrt{23 - 4\sqrt{33}}$	$7 - 4\sqrt{3}$
∞	0	0

Table 1: The parameter α for some half-integer values of m .

vanishes when $0 < 2m - 1 < 2m + 2$. In the tetrahedral case,

$$\det \Psi_1 = 4(1 - 3\alpha^2)^2(3 - \alpha^2)^2$$

and hence $\alpha \neq \frac{1}{\sqrt{3}}$ for $m > \frac{1}{2}$. But α tends to zero as $m \rightarrow \infty$, and α begins at $\sqrt{3}$ when $m = 0$, so by continuity, $\alpha = \frac{1}{\sqrt{3}}$ for some value of m . In fact,

$$\alpha = \frac{1}{\sqrt{3}} \Leftrightarrow m = \frac{1}{2},$$

since $H^0(\Sigma, L^{2m-1}(k-2, 0)) = H^0(\Sigma, \mathcal{O}(k-2, 0))$ has nontrivial sections, and thus Ψ_1 has nontrivial kernel, so $\det \Psi_1 = 0$.

The preceding calculation is sufficient for our purposes to get the relation (67) once we have found a suitable path γ on E from q to p , but we need at least one more value of (α, m) to be able to find γ , as explained in section 6. However, similar calculations on $H^0(\Sigma, L^{2m-2}(k-2, 0)) = H^0(\Sigma, \mathcal{O}(-4, k+2))$ yield a map Ψ_2 with $\det \Psi_2 = 4(\alpha^2 + 5)^2(1 - 3\alpha^2)^2(\alpha^2 - 4\alpha + 1)^2(\alpha^2 + 4\alpha + 1)^2$, and this enables one to conclude that $\alpha = 2 - \sqrt{3}$ when $m = 1$. This technique applies to yield an algebraic value of α for any half-integer mass m ; Table 1 summarises these results for $m \leq \frac{3}{2}$.

In the octahedral case,

$$\det \Psi_1 = 16(1 + 5\alpha)^2(\alpha + 5)^3(3\alpha - 1)^3(\alpha - 1)^4$$

and using the argument above we deduce that

$$\alpha = \frac{1}{3} \Leftrightarrow m = \frac{1}{2}. \tag{80}$$

The next values for α , found by computing the determinants up to $m = \frac{3}{2}$, are displayed in Table 1.

10 The general mass constraints

We start by constructing the quotients π in (66) explicitly, and in particular a realisation of the elliptic curves E , by means of invariant theory. To do this, we use the procedure illustrated in Example 5.3 to find rational invariants $\hat{v}, \hat{x}, \hat{y} \in \mathbb{C}(\Sigma)^{A_4}$ given in the coordinates

of $\mathbb{P}^1 \times \mathbb{P}^1$ by

$$\begin{aligned}\hat{v} &= \frac{P_3}{P_1^3} := \frac{(w+z)((wz)^2-1)}{(w-z)^3}, \\ \hat{x} &= \frac{P_4}{P_1^4} := \frac{w^4z^4 + w^4 + z^4 + 12w^2z^2 + 1}{(w-z)^4}, \\ \hat{y} &= \frac{P_6}{P_1^6} := \frac{(wz)^6 - ((wz)^2+1)((w+z)^4 + 4wz(w+z)^2 + (wz)^2) + 1}{(w-z)^6}.\end{aligned}$$

Here, P_ℓ denote forms of degree (ℓ, ℓ) that are projectively invariant under (59). They satisfy a relation (in degree 12) which can be written in terms of the invariants above as

$$4\hat{x}^3 - 11\hat{x}^2 - 4\hat{y}^2 - 14\hat{v}^2 - 27\hat{v}^4 + 2\hat{x}(5 + 9\hat{v}^2) - 3 = 0. \quad (81)$$

Out of these rational functions, we produce maps into \mathbb{P}^2 whose restriction to each Σ has A_4 -orbits as fibres. The images are determined by a single relation among the invariants, which can be used to describe the elliptic curves Σ/A_4 . This procedure is easily adapted to the octahedral case, as we shall explain below. In practice, it is convenient to choose the coordinates of the embeddings to obtain an image in standard Weierstraß form.

10.1 $k = 3$ tetrahedral symmetry

Using equation (60) to eliminate $\hat{v} = \frac{i}{\alpha}$, we obtain from (81) the relation on Σ

$$4\hat{y}^2 = 4\hat{x}^3 - 11\hat{x}^2 + \left(10 - \frac{18}{\alpha^2}\right)\hat{x} - 3 + \frac{14}{\alpha^2} - \frac{27}{\alpha^4}.$$

So we redefine the invariants as $x := \hat{x} - \frac{11}{12}$, $y := 2\hat{y}$, in order to obtain the standard plane cubic

$$y^2 = 4x^3 - g_2x - g_3 =: F(x), \quad (82)$$

where

$$g_2 = \frac{1}{12} + \frac{18}{\alpha^2}, \quad g_3 = -\frac{1}{216} + \frac{5}{2\alpha^2} + \frac{27}{\alpha^4}. \quad (83)$$

The picture is that $\pi := [1 : x : y]$ maps A_4 -orbits in Σ (given by equation (60)) into \mathbb{P}^2 , and the image E is realised as the elliptic curve given by (82). From this equation, we read off that $x \circ u^{-1} = \wp$ is a Weierstraß function of E and $y \circ u^{-1} = \pm \wp'$ its derivative up to sign, where $u^{-1} : \mathbb{C} \rightarrow E$ is the well-defined inverse to a “uniformisation map”. We use e_1, e_2, e_3 to denote the zeroes of F ,

$$F(x) = 4(x - e_1)(x - e_2)(x - e_3)$$

with $e_1 + e_2 + e_3 = 0$. From (83), we compute the j -invariant of E to be

$$j(\alpha) = \frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{\alpha^2(\alpha^2 + 216)^3}{2^6 3^3(\alpha^2 - 27)^3}.$$

One way of realising E is as a degree 3 branched cover $[1 : x \circ \pi^{-1}] : E \rightarrow \mathbb{P}^1$, with branch points $[1 : e_1], [1 : e_2]$ and $[1 : e_3]$ and suitable branch cuts. Clearly, two zeroes of F must be nonreal and complex conjugates, while the other (say e_2) is real and positive since $g_3 = 4e_1e_2e_3 > 0$. We take $\text{Im}(e_1) > 0$. Notice that $\text{Re } e_1 = \text{Re } e_3 = -\frac{1}{2}e_2$ and $\text{Im } e_3 = -\text{Im } e_1$.

We must define branch cuts, the sheet decomposition, a holomorphic 1-form ω and a basis of 1-homology for E before we can compute the integrals in the relation (71). We consider the standard holomorphic 1-form on E given by

$$\omega = \pm \frac{dx}{\sqrt{F(x)}}. \quad (84)$$

up to sign. The conventions that we shall follow are illustrated in Figure 2. One branch cut runs from e_2 to ∞ downwards while the other one joins e_1 to e_3 avoiding the negative real axis; these separate E into two sheets (i) and (ii). The poles of $\frac{dw}{w} - \frac{dz}{z}$ are easily seen to be

$$\begin{aligned} p_1 &= (\bar{\rho}\sqrt{\alpha}, 0), & p_2 &= (-\bar{\rho}\sqrt{\alpha}, 0), \\ q_1 &= (0, \rho\sqrt{\alpha}), & q_2 &= (0, -\rho\sqrt{\alpha}), \end{aligned}$$

where $\rho := e^{\pi i/4}$. Their images on $E \subset \mathbb{P}^2$ can be computed as

$$\begin{aligned} p &:= \pi(p_1) = \left[1 : \frac{1}{12} - \frac{1}{\alpha^2} : -\frac{2i}{\alpha} \left(1 + \frac{1}{\alpha^2} \right) \right] = \pi(p_2), \\ q &:= \pi(q_1) = \left[1 : \frac{1}{12} - \frac{1}{\alpha^2} : +\frac{2i}{\alpha} \left(1 + \frac{1}{\alpha^2} \right) \right] = \pi(q_2), \end{aligned}$$

and they lie on the same fibre of the double cover $[1 : x] : E \rightarrow \mathbb{P}^1$, as we claimed before. Since $\alpha < \sqrt{3}$ implies $\frac{1}{12} - \frac{1}{\alpha^2} < 0$, the common projection of both p and q onto the x -plane lies on the negative real axis. As before, paths are drawn with continuous or dotted lines according to whether they lie on sheets (i) or (ii), and we have defined a, b and the path from p to q (avoiding a and b) following this notation; in particular, q lies on sheet (i) and p on sheet (ii). We choose the signs in (84) so that the coefficient at p has positive imaginary part.

Equation (71) relates the periods

$$2\varpi := \oint_a \omega = 2 \int_{e_2}^{+\infty} \frac{dx}{\sqrt{F(x)}} \quad (85)$$

$$2\varpi' := \oint_b \omega = -2i \int_{e_2}^{e_1} \frac{dx}{\sqrt{-F(x)}} \quad (86)$$

to the integral

$$\int_p^q \omega = 2i \int_{\frac{1}{12} - \frac{1}{\alpha^2}}^{-\infty} \frac{dx}{\sqrt{-F(x)}} \quad (87)$$

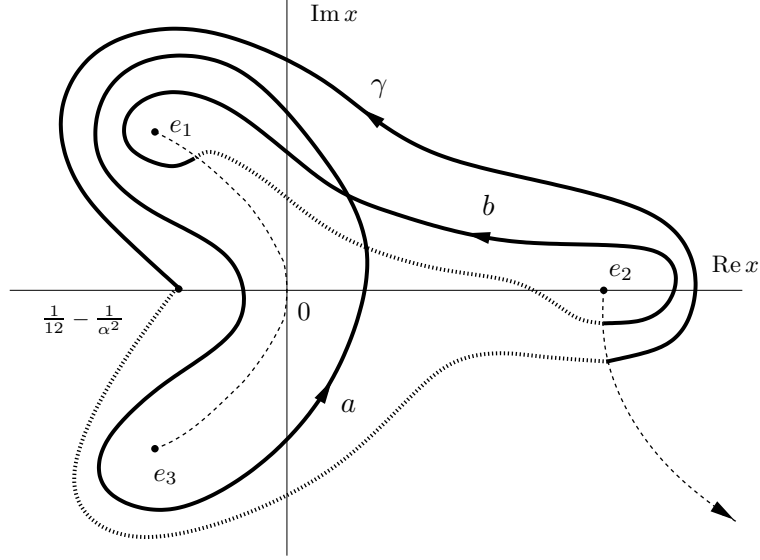


Figure 2: Tetrahedrally symmetric 3-monopole: contours of integration on $E = \Sigma/A_4$.

as

$$\ell_1 \varpi + \ell_2 \varpi' = 2i(2m+3) \int_{\frac{1}{12} - \frac{1}{\alpha^2}}^{-\infty} \frac{dx}{\sqrt{-F(x)}}. \quad (88)$$

Notice that ϖ is real, while the integral (87) is pure imaginary, and that one has

$$\begin{aligned} 2 \operatorname{Re} \varpi' &= -i \int_{e_2}^{e_1} \frac{dx}{\sqrt{-F(x)}} + i \overline{\int_{e_2}^{e_1} \frac{dx}{\sqrt{-F(x)}}} \\ &= -i \int_{e_1}^{e_2} \frac{dx}{\sqrt{-F(x)}} + i \int_{e_2}^{e_3} \frac{dx}{\sqrt{-F(x)}} \\ &= i \int_{e_1}^{e_3} \frac{dx}{\sqrt{-F(x)}} \\ &= \varpi. \end{aligned}$$

Taking real parts in both sides of equation (88), one then obtains the relation

$$\ell_2 = -2\ell_1. \quad (89)$$

These integrals are most conveniently dealt with using uniformisation. The standard uniformisation point is the set of 2-poles of \wp , thus of 2-poles of x :

$$x = \infty \Leftrightarrow w = z;$$

an easy check gives that all the six points of $\Sigma \cap \mathbb{P}_\Delta^1$ are poles, and they are mapped to $[0 : 0 : 1]$ by π , the branch point at infinity of $[1 : x \circ \pi^{-1}] : E \rightarrow \mathbb{P}^1$. One has $\wp(\varpi) = e_2$

and $\wp(\varpi') = e_3$, so to conform with standard practice we should use the half-periods

$$\begin{aligned}\varpi_2 &= \varpi, \\ \varpi' &= \frac{\varpi_1 + \varpi_2}{2} \Rightarrow \varpi_1 = 2\varpi' - \varpi.\end{aligned}\tag{90}$$

Now we can write formally

$$\begin{aligned}i \int_{\frac{1}{12} - \frac{1}{\alpha^2}}^{-\infty} \frac{dx}{\sqrt{-F(x)}} &= \int_{\wp \circ \wp^{-1}(\frac{1}{12} - \frac{1}{\alpha^2})}^{\wp(0)} \frac{d\wp}{\wp'} \\ &= \int_{\wp^{-1}(\frac{1}{12} - \frac{1}{\alpha^2})}^0 du \\ &= \mp \wp^{-1} \left(\frac{1}{12} - \frac{1}{\alpha^2} \right),\end{aligned}\tag{91}$$

where \wp^{-1} is determined only up to sign (as \wp has order 2 in a fundamental region) and up a point in the lattice $\Lambda = 2\varpi_1\mathbb{Z} \oplus 2\varpi_2\mathbb{Z}$ inside the u -plane. Equation (88) then gives, making use of (89),

$$\frac{(\varpi - 2\varpi')\ell_1}{2(2m+3)} \equiv \pm \wp^{-1} \left(\frac{1}{12} - \frac{1}{\alpha^2} \right) \pmod{\Lambda}.$$

The ambiguity introduced in (91) is removed once we apply \wp to both sides of this equation:

$$\wp \left(\frac{\varpi_1 \ell_1}{2(2m+3)} \right) = \frac{1}{12} - \frac{1}{\alpha^2},$$

The only information missing is the integer ℓ_1 determining $c = \ell_1(1, -2) \in H_1(E; \mathbb{Z})$, but this can be obtained directly from (88) and (89) using the data (α, m) we calculated in section 9. We always find

$$\ell_1 = 4$$

by evaluating the integrals numerically for all the positive values of m in Table 1. This confirms that we are working with a path γ on E that is the image of a path with zero intersection with the 1-cycles of Σ . We end up with equation (67), and this completes the proof of Theorem 6.1.

10.2 $k = 4$ octahedral symmetry

When (62) is used to eliminate $\hat{x} = 1 - (1 + i\beta\hat{v})/\alpha$, one obtains the relation on Σ

$$\begin{aligned}-4\hat{y}^2 - 27\hat{v}^4 + \frac{2i\beta}{\alpha} \left(\frac{2\beta^2}{\alpha^2} - 9 \right) \hat{v}^3 + \left(4 - \frac{18}{\alpha} + \left(\frac{12}{\alpha} - 1 \right) \frac{\beta^2}{\alpha^2} \right) \hat{v}^2 + \\ \frac{2i\beta}{\alpha^2} \left(1 - \frac{6}{\alpha} \right) \hat{v} + \frac{1}{\alpha^2} \left(1 - \frac{4}{\alpha} \right) = 0.\end{aligned}\tag{92}$$

In principle, one could follow the same procedure as in the $k = 3$ case to obtain the relation among α, β and m , but a calculation for general values of the parameters is out of hand

as one now must deal with the roots of a quartic polynomial, and these then give rise to a less symmetric form of the Weierstraß equation (82). In the following, we shall only consider the case of octahedral symmetry $\beta = 0$, working with Σ/S_4 rather than Σ/A_4 , which considerably simplifies the calculations as one should expect.

The extension from tetrahedral to octahedral symmetry can be realised by adding the rotation (63), which changes the tetrahedral invariants as

$$\hat{v} \mapsto -\hat{v}, \quad \hat{x} \mapsto \hat{x}, \quad \hat{y} \mapsto -\hat{y}.$$

Therefore,

$$\hat{x}, \quad \hat{v}\hat{y}, \quad \text{and} \quad \hat{v}^2$$

are octahedral invariants, and a cubic relation among them on Σ can be obtained simply by multiplying both sides of (92) by \hat{v}^2 . This relation is brought to Weierstraß form (82) if one uses the coordinates

$$y := \sqrt{2}i\hat{v}\hat{y}, \quad x := \frac{3}{2}\hat{v}^2 + \frac{1}{3\alpha} - \frac{2}{27};$$

the elliptic invariants are now

$$g_2 = \frac{16}{253} - \frac{16}{27\alpha} + \frac{5}{3\alpha^2} - \frac{4}{3\alpha^3} \quad (93)$$

and

$$g_3 = \frac{64}{19683} - \frac{32}{729\alpha} + \frac{2}{9\alpha^2} - \frac{41}{81\alpha^3} + \frac{4}{9\alpha^4}. \quad (94)$$

These yield the j -invariant

$$j(\alpha) = \frac{(432\alpha^3 - 4048\alpha^2 + 11385\alpha - 9108)^3}{2^2 3^3 11^3 23^3 (\alpha - 4)^2 (\alpha - 3)^3}.$$

Again, we realise $E := \Sigma/S_4$ as the image of the map $\pi := [1 : x : y]$ to \mathbb{P}^2 . It is easy to see that $F(x)$ has again a real positive zero e_2 and two distinct complex conjugate zeroes e_1, e_3 ; we choose the convention $\text{Im } e_1 > 0$ as before.

The differential $\frac{dw}{w} - \frac{dz}{z}$ on Σ has four poles with positive residue and four poles with negative residue, and their images on $E \subset \mathbb{P}^2$ are, respectively,

$$p = \left[1 : -\frac{2}{27} - \frac{7}{6\alpha} : \sqrt{2}i \frac{1+\alpha}{\alpha^2} \right],$$

$$q = \left[1 : -\frac{2}{27} - \frac{7}{6\alpha} : -\sqrt{2}i \frac{1+\alpha}{\alpha^2} \right].$$

We should integrate on Σ along four paths, from $\pi^{-1}(p)$ to $\pi^{-1}(q)$ points, which do not cross a basis of 1-cycles of Σ . Contrary to the $k = 3$ case, it will turn out that the image of such paths will have nontrivial intersection with a standard basis of E . We follow the same conventions as above for the covering $[1 : x] : E \rightarrow \mathbb{P}^1$ and the periods (85)–(86), but not for the path γ , which we define as having $\# \langle \gamma, b \rangle = +1$ as illustrated in Figure 3.

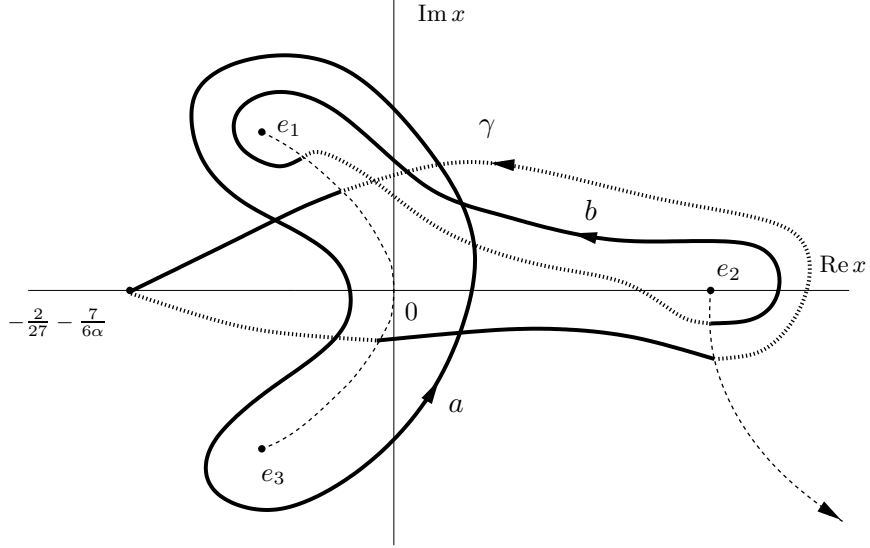


Figure 3: Octahedrally symmetric 4-monopole: contours of integration on $E = \Sigma/S_4$.

We obtain the relation for ω and c as in (84) and (70),

$$\oint_c \omega = 8(m+2) \int_p^q \omega,$$

and using our conventions we find

$$\ell_1 \varpi + \ell_2 \varpi' = 4i(m+2) \int_{e_2}^{-\frac{2}{27} - \frac{7}{6\alpha}} \frac{dx}{\sqrt{-F(x)}}. \quad (95)$$

Again, (89) holds, and we are led to the relation

$$\wp \left(\frac{\varpi_1 \ell_1}{2(m+2)} \right) = \frac{-4\alpha^4 + 10\alpha^3 - 115\alpha^2 + 60\alpha - 3}{54\alpha^2(\alpha+1)^2},$$

where we made use of the duplication formula for \wp . Given (80), we find from (95)

$$\ell_1 = 6,$$

and the same result is obtained for the other (α, m) data, which confirms that the right path γ has been chosen. This completes the proof of Theorem 6.2.

11 Rational and infinite mass

This final section illustrates how one may deal with the general mass constraints (67) and (68) in two important concrete cases, providing nontrivial checks to our calculations.

11.1 Solutions for rational mass

We shall now explain briefly how to solve the mass conditions (67) and (68) explicitly to calculate α for rational values of the monopole mass. We shall be making use of the fact that, for $m \in \mathbb{Q}$, the left-hand side of either of these conditions is a division value of the \wp -function, i.e., a complex number of the form

$$\wp\left(\frac{2k_1\varpi_1 + 2k_2\varpi_2}{n}\right)$$

where $n \in \mathbb{N}$ and $k_1, k_2 \in \mathbb{Z}$ are such that $(k_1, k_2) \not\equiv (0, 0) \pmod{n}$. Classical techniques in the theory of elliptic functions [19] can be used to show that division values correspond exactly to the roots of the so-called *special division equation* $P_n(\wp(u)) = 0$, where $P_n(\wp)$ is a polynomial of degree $\frac{n^2-1}{2}$ or $\frac{n^2-4}{2}$ defined by

$$P_{2\ell+1}(\wp(u)) := \frac{\psi_{2\ell+1}(u)}{2\ell+1} \quad \text{or} \quad P_{2\ell+2}(\wp(u)) := -\frac{\psi_{2\ell+2}(u)}{(\ell+1)\wp'(u)}$$

according to whether n is odd or even; here,

$$\psi_n(u) := \frac{(n-1)^{n-1}}{\left(\prod_{j=1}^{n-1} j!\right)^2} \begin{vmatrix} \wp'(u) & \wp''(u) & \cdots & \wp^{(n-1)}(u) \\ \wp''(u) & \wp'''(u) & \cdots & \wp^{(n)}(u) \\ \vdots & \vdots & \ddots & \vdots \\ \wp^{(n-1)}(u) & \wp^{(n)}(u) & \cdots & \wp^{(2n-3)}(u) \end{vmatrix}$$

is easily seen to satisfy $\psi_n(-u) = (-1)^{n+1}\psi_n(u)$, so indeed $P_n(\wp) \in \mathbb{Q}[g_2, g_3][\wp]$.

To calculate α for $m \in \mathbb{Q}$, one replaces $\wp(u)$ by the right-hand side of (67) or (68) in the special division equation for the corresponding elliptic curve $E = \Sigma/G$, thus obtaining a polynomial equation for α ; by inspection, the minimal polynomial for α is recovered as one of the factors, and hence α itself. This argument shows that, for $m \in \mathbb{Q}$, equations (67) and (68) determine α as an algebraic number.

The results of Table 1 in section 9 are easily checked using this procedure; in particular, we can compute $\lim_{m \rightarrow 0} \alpha$ and write the limit “nullaron” curves in the form (51), from which we can recover the corresponding rational maps. For $m \geq 2$, the algorithm in section 9 becomes impractical, whereas the special division equation can still be used to generate minimal polynomials for the parameter α . In addition, one can calculate in principle the value of α for any $m \in \mathbb{Q}$. For example, in the $k = 3$, $G = A_4$ case, for $m = \frac{1}{3}$, we obtain the minimal polynomial

$$\alpha^{10} - 39\alpha^8 + 506\alpha^6 + 866\alpha^4 - 715\alpha^2 - 11,$$

which yields $\alpha \simeq 0.791875$ as the unique positive real root larger than $\alpha|_{m=\frac{1}{2}} = \frac{1}{\sqrt{3}}$.

11.2 Euclidean limit

As explained in section 4.4, centred spectral curves degenerate to k copies of the diagonal \mathbb{P}^1_Δ when the limit $m \rightarrow \infty$ is taken, but a rescaling of the metric yields curves in $T\mathbb{P}^1$

which can be interpreted as spectral curves of euclidean monopoles. This limit process allows one to derive the spectral curves of platonic monopoles in euclidean space from our results for hyperbolic monopoles; it will also provide a nontrivial check on our calculations.

We focus on the case $k = 3$, $G = A_4$ for brevity. Given (52), we expect the rescaled parameter $\tilde{\alpha} := m^3\alpha$ to tend to a finite limit as $m \rightarrow \infty$ (and $\alpha \rightarrow 0$), and our aim is to calculate this limit using the relation (67). The natural way to proceed is to rescale the A_4 -invariants on Σ in such a way that the quotient elliptic curve $E = \Sigma/A_4 \subset \mathbb{P}^2$ has a sensible limit; thus we use $\tilde{x} := x/m^4$ and $\tilde{y} := y/m^6$ to embed E in \mathbb{P}^2 . When $m \rightarrow \infty$, one obtains the Weierstraß equation for E

$$\tilde{y}^2 = 4\tilde{x}^3 - \frac{27}{\tilde{\alpha}^2} =: 4(\tilde{x} - \tilde{e}_1)(\tilde{x} - \tilde{e}_2)(\tilde{x} - \tilde{e}_3) =: \tilde{F}(\tilde{x}),$$

where we use conventions for the invariants (and periods) consistent with section 10.1. This is the same elliptic curve as in the calculations in section 9 of [14], since the j -invariant is zero in both cases. Clearly, the periods transform as $\tilde{\omega}_j = m^2\omega_j$ under our rescaling. Using the asymptotics $\wp(\rho) = \frac{1}{\rho^2} + O(\rho^2)$ as $\rho \rightarrow 0$, one obtains from (67) the relation

$$\frac{m^6}{\tilde{\omega}_1^2} + O\left(\frac{1}{m^6}\right) = \frac{1}{12} - \frac{m^6}{\tilde{\alpha}^2} \xrightarrow{m \rightarrow \infty} \tilde{\omega}_1 = i\tilde{\alpha} \quad (96)$$

for the limit elliptic curve. Now we evaluate, in the limit,

$$\begin{aligned} \tilde{\omega}_1 &= -2i \int_{\tilde{e}_2}^{\tilde{e}_1} \frac{d\tilde{x}}{\sqrt{-\tilde{F}(\tilde{x})}} - \int_{\tilde{e}_2}^{+\infty} \frac{d\tilde{x}}{\sqrt{\tilde{F}(\tilde{x})}} \\ &= -\frac{2^{1/3}e^{2\pi i/3}\sqrt{\pi}\tilde{\alpha}^{2/3}\Gamma\left(\frac{1}{3}\right)}{3\Gamma\left(\frac{5}{6}\right)} - \frac{2^{1/3}\sqrt{\pi}\tilde{\alpha}^{2/3}\Gamma\left(\frac{1}{6}\right)}{9\sqrt{3}\Gamma\left(\frac{5}{3}\right)}. \end{aligned} \quad (97)$$

Equating the right-hand sides of (96) and (97), we find

$$\tilde{\alpha} = \frac{\Gamma\left(\frac{1}{3}\right)^9}{2^6\pi^3};$$

thus the limit curve in $T\mathbb{P}^1$

$$\eta^3 + 2i\tilde{\alpha}\zeta(\zeta^4 - 1) = 0$$

is precisely the spectral curve of a euclidean 3-monopole with tetrahedral symmetry as found in [14, 15].

Acknowledgements

The authors are grateful to Michael Murray for discussions and support. The second author's work is financed by the Australian Research Council, and he would like to thank the Department of Mathematics and Statistics of the University of Melbourne for hospitality.

References

- [1] M. ADLER and P. VAN MOERBEKE: Linearization of Hamiltonian systems, Jacobi varieties and representation theory. *Adv. in Math.* **38** (1980) 318–379
- [2] E. ARBARELLO, M. CORNALBA, P. GRIFFITHS, J. HARRIS: *Geometry of Algebraic Curves*, Vol. I. Grundlehren der Mathematischen Wissenschaften 267, Springer-Verlag, 1985
- [3] M.F. ATIYAH: Magnetic monopoles in hyperbolic spaces. In *Vector Bundles in Algebraic Geometry (Bombay Colloquium 1984)*, Tata Institute. Oxford University Press, pp. 1–33
- [4] M.F. ATIYAH: Magnetic monopoles and the Yang–Baxter equations. *Int. J. Mod. Phys. A* **6** (1991) 2761–2774
- [5] M.F. ATIYAH and N.J. HITCHIN: *The Geometry and Dynamics of Magnetic Monopoles*. Princeton University Press, 1988
- [6] M.F. ATIYAH and M.K. MURRAY: Monopoles and Yang–Baxter equations. In *Further Advances in Twistor Theory, vol. II*, eds. L.J. Mason, L.P. Hughston and P.Z. Kobak. Longman, 1995, pp. 13–14
- [7] P.F. BYRD and M.D. FRIEDMAN: *Handbook of Elliptic Integrals for Engineers and Physicists*. Springer-Verlag, 1954
- [8] E. ERCOLANI and A. SINHA: Monopoles and Baker functions. *Commun. Math. Phys.* **125** (1989) 385–416
- [9] R. FRICKE: Elliptische Funktionen. In *Encyklopädie der mathematischen Wissenschaften* II 2, eds. H. Burkhardt, W. Wirtinger, R. Fricke and E. Hilb, Teubner, Leipzig, 1913, pp. 177–348
- [10] P. GRIFFITHS: Linearizing flows and a cohomological interpretation of Lax equations. *Amer. J. Math.* **107** (1985), 1445–1484
- [11] P. GRIFFITHS and J. HARRIS: *Principles of Algebraic Geometry*. Wiley, 1978
- [12] N.J. HITCHIN: Monopoles and geodesics. *Commun. Math. Phys.* **83** (1982) 579–602
- [13] N.J. HITCHIN: Magnetic monopoles with Platonic symmetry. In *Moduli of Vector Bundles (Sanda and Kyoto, 1994)*, ed. M. Maruyama. Lecture Notes in Pure and Applied Mathematics 179, Springer-Verlag, 1996, pp. 55–63
- [14] N.J. HITCHIN, N.S. MANTON and M.K. MURRAY: Symmetric monopoles. *Nonlinearity* **8** (1995) 661–692, [dg-ga/9503016](#)
- [15] C.J. HOUGHTON, N.S. MANTON and N.M. ROMÃO: On the constraints defining BPS monopoles. *Commun. Math. Phys.* **212** (2000) 219–243, [hep-th/9909168](#)
- [16] C.J. HOUGHTON and P.M. SUTCLIFFE: Tetrahedral and cubic monopoles. *Commun. Math. Phys.* **180** (1996) 343–361, [hep-th/9601146](#)
- [17] J. HURTUBISE: $SU(2)$ monopoles of charge 2. *Commun. Math. Phys.* **92** (1983) 195–202
- [18] S. JARVIS and P. NORBURY: Zero and infinite curvature limits of hyperbolic monopoles. *Bull. London Math. Soc.* **29** (1997) 737–744
- [19] L. KIEPERT: Wirkliche Ausführung der ganzzahligen Multiplication der elliptischen Functionen. *J. reine angew. Math.* **76** (1872) 21–33

- [20] F. KLEIN: Vorlesungen über das Ikosaeder und die Auflösung der Gleichung vom fünften Grade. Teubner, Leipzig, 1884
- [21] J. MALDACENA: The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200](#)
- [22] N.S. MANTON: A remark on the scattering of BPS monopoles. *Phys. Lett. B* **110** (1982) 54–56
- [23] N.S. MANTON and P.M. SUTCLIFFE: Topological Solitons. Cambridge University Press, 2004
- [24] M.K. MURRAY, P. NORBURY and M.A. SINGER: Hyperbolic monopoles and holomorphic spheres. *Ann. Global Anal. Geom.* **23** (2003) 101–128, [math.DG/0111202](#)
- [25] M. MURRAY and M. SINGER: Spectral curves of non-integral hyperbolic monopoles. *Nonlinearity* **9** (1996) 973–997
- [26] P. NORBURY: Boundary algebras of hyperbolic monopoles. *J. Geom. Phys.* **51** (2004) 13–33, [math-ph/0106013](#)
- [27] G.B. SEGAL and A. SELBY: The cohomology of the space of magnetic monopoles. *Commun. Math. Phys.* **177** (1996) 775–787
- [28] A. SEN: Dyon-monopole bound states, self-dual harmonic forms on the multi-monopole moduli space, and $SL(2, \mathbb{Z})$ -invariance of string theory. *Phys. Lett. B* **329** (1994) 217–221, [hep-th/9402032](#)
- [29] P.M. SUTCLIFFE: Monopole zeros. *Phys. Lett. B* **376** (1996) 103–110, [hep-th/9603065](#)